

Multilinear Map

Let F be a field and let V_1, \dots, V_m be vector spaces over it. A map $f : V_1 \times \dots \times V_m \rightarrow F$ is called **multilinear** if, for any fixed $v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_m$, the function is linear with respect to the i^{th} component.

Often times each $V_i = V$. When $m = 2$ and $V_1 = V_2 = V$, and V has basis e_1, \dots, e_n , for example, we can consider

$$B\left(\sum a_i e_i, \sum b_j e_j\right) = \sum_{i,j} a_i b_j B(e_i, e_j)$$

or correspondingly the **Gram matrix** $G = \{B(e_i, e_j)\}_{i,j=1}^n$ of B (with respect to that basis):

$$B\left(\sum a_i e_i, \sum b_j e_j\right) = (a_1, \dots, a_n) G (b_1, \dots, b_n)^T.$$

In particular if $G = I$ then B simply outputs the dot product.

B is called a **symmetric bilinear form** if $B(v_1, v_2) = B(v_2, v_1)$. B is called an **alternating bilinear form** if $B(v, v) = 0$ for all v . (This corresponds to Gram matrix being skew symmetric.)

Remark. Intuitively we want alternating bilinear forms to satisfy $B(v_1, v_2) = -B(v_2, v_1)$, but in a field with characteristic 2, $-1 = 1$, and this is no good. However, the definition above does imply what we want:

$$0 = B(v_1 + v_2, v_1 + v_2) = B(v_1, v_1) + B(v_1, v_2) + B(v_2, v_1) + B(v_2, v_2) = B(v_1, v_2) + B(v_2, v_1).$$

Therefore, if $\text{char}(F) \neq 2$ these two conditions are equivalent.

Also, more formally A is skew symmetric if $a_{i,j} = -a_{j,i}$ and $a_{i,i} = 0$. If $\text{char}(F) = 2$ this implies A is symmetric.

We say f is **m -linear** on $V \times \dots \times V$ (m times) if the order of its inputs doesn't matter. Similarly, we say f is an **alternating m -linear form** if $f(v_1, \dots, v_n) = 0$ if $v_i = v_j$ for some $i \neq j$.

Remark. If $m \geq \dim(V) = n$ then the only alternating m -linear form is the zero function.

We use $\Lambda^k(V)$ to denote the space of k -linear alternating forms on V . An immediate result is

$$\dim \Lambda^k(V) = \binom{n}{k}$$

(for we have k ways to choose a basis), and in particular

$$\dim \Lambda^n(V) = 1 \quad \text{and} \quad \dim \Lambda^k(V) = 0 \text{ for } k \geq n.$$

Just like before, by multi-linearity,

$$\begin{aligned} f\left(\sum a_{1,i} e_i, \sum a_{2,i} e_i, \dots, \sum a_{n,i} e_i\right) &= \sum a_{1,i_1} a_{2,i_2} \dots a_{n,i_n} f(e_{i_1} e_{i_2} \dots e_{i_n}) \\ &= \sum_{i_j \text{ distinct}} f(e_{i_1} e_{i_2} \dots e_{i_n}) \\ &= \sum_{\sigma \in S_n} \text{sgn}(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} \dots a_{n,\sigma(n)} f(e_1, \dots, e_n). \end{aligned}$$

Determinant

Let A be a $n \times n$ matrix over F . We view a as an n -tuple of row vectors, namely $a = (r_1, \dots, r_n)^t$ where r_i is the i^{th} row. Under this context, we define the determinant to be the unique(!) alternating n -linear form on the (n -dimensional) space of row vectors in \mathbb{R}^n as $\det(I) = 1$.

Immediately following this definition and the previous computations, we see

$$\det(\{a_{i,j}\}) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) a_{1,\sigma(1)} \dots a_{n,\sigma(n)},$$

namely the “brute force” way to compute a determinant. Following this result, we have $\det(AB) = \det(A)\det(B)$. (Proof sketch of $\det(AB) = \det(A)\det(B)$: fix B and define $f_B(A) := \det(AB)$. Check this is an alternating n -linear form on the row space, so $f_B(A) = \lambda \det(A)$ for some λ . Letting $A = I$ we get the result.)

We also obtain many standard facts of matrices, for example

- $\det A = \det A^T$ (for A^T just sums over $\sigma^{-1} \in S_n$ and $\text{sgn}(\sigma) = \text{sgn}(\sigma^{-1})$);
- $\det A \neq 0$ iff $\text{rank}(A) = n$ iff A is invertible;
- $\det A =$ product of diagonal entries for diagonal/triangular matrices;
- To compute the determinant of $n \times n$ matrix, we can compute the determinant of $n(n-1) \times (n-1)$ matrices and with some simple addition and multiplication;