Solution to 3.2. The function corresponding to the first graph is clearly not (semi)concave since the superlevel sets are not even simply connected. It is also not convex since there are certain lines that intersect the level sets in a way such that the distance between points corresponding to 1 and 2 is shorter than that between 2 and 3, i.e., the function increases at at a decreasing rate. It may be quasiconvex (i.e., there is not enough info to reject this).

For the second one, assuming the graph actually represents what the contour plot looks like (i.e., no weird shape in other areas), this function cannot be convex as its sublevel sets are not simply connected. It may or may not be (quasi)concave.

Solution to 3.16. (1) This one is obviously convex. It is monotone so it is both quasiconvex and quasiconcave.

(2) The Hessian is I^T which is neither PSD or NSD so f is not convex or concave. The superlevel sets for f and $-f$ are

$$\{(x_1, x_2) \in \mathbb{R}^2_{++} : x_1 x_2 \geqslant k\} \qquad \text{and} \qquad \{(x_1, x_2) \in \mathbb{R}^2_{++} : x_1 x_2 \leqslant k\}.$$

The first is convex: with the assumption that x, y are in the first quadrant, the area is above $y = k/x$ and is therefore convex. The other is the complement with respect to $\mathbb{R}^2_{++}$ and is certainly not convex. Hence f is quasiconvex but not quasiconcave.

(3) From (2), we see the superlevel set of $1/(x_1 x_2)$ corresponds to "some" (i.e., taking reciprocal) superlevel set of $-x_1 x_2$, so the superlevel set is not convex and f is not quasiconvave, not to mention concave.

On the other hand f is convex: the Hessian is

$$\frac{1}{x_1 x_2}\begin{bmatrix} 2/x_1^2 & 1/(x_1 x_2) \\ 1/(x_1 x_2) & 2/x_2^2 \end{bmatrix}$$

and

$$\begin{bmatrix} y_1 & y_2 \end{bmatrix}\begin{bmatrix} 2/x_1^2 & 1/(x_1 x_2) \\ 1/(x_1 x_2) & 2/x_2^2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{2y_1^2}{x_1^2} + \frac{2y_2^2}{x_2^2} + \frac{2y_1 y_2}{x_1 x_2} > 0$$

unless $y_1 = y_2 = 0$, so the Hessian is positive definite, i.e., f is convex and quasiconvex.

(4) The Hessian is

$$\begin{bmatrix} 0 & -1/x_2^2 \\ -1/x_2^2 & 2x_1/x_2^3 \end{bmatrix}$$

and

$$\begin{bmatrix} y_1 & y_2 \end{bmatrix}\begin{bmatrix} 0 & -1/x_2^2 \\ -1/x_2^2 & 2x_1/x_2^3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{2x_1 y_2^2}{x_2^3} - 2\frac{y_2^2}{x_2^2}.$$

Let $x_2 = 1$. If $x_1 < 1$ then above < 0 and if $x_1 > 1$ then above > 0. That is, the Hessian is neither PSD nor NSD so f is not concave or convex. It is certainly quasiconvex and quasiconcave since the sublevel and superlevel sets are characterized by $x_1 \geqslant t x_2$ or $x_1 \leqslant t x_2$, i.e, half spaces!

(5) The Hessian is

$$\begin{bmatrix} 2/x_2 & -2x_1 x_2^2 \\ -2x_1/x_2^2 & 2x_1^2/x_2^3 \end{bmatrix} \sim \begin{bmatrix} 1 & -x_1/x_2 \\ -x_1/x_2 & x_1^2/2x^2 \end{bmatrix}$$

and

$$\begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 1 & -x_1/x_2 \\ -x_1/x_2 & x_1^2/x_2^2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = y_1^2 - 2\frac{x_1 y_1}{x_2} + \frac{x_1^2 y_2^2}{x_2^2} > 0$$

unless $y_1 = y_2 = 0$, so f is convex and quasiconvex but not concave or quasiconcave.

(6) This is just pure algebra... The Hessian is

$$\begin{bmatrix} \alpha(\alpha-1)x_1^{\alpha-2}x_2^{1-\alpha} & \alpha(1-\alpha)x_1^{\alpha-1}x_2^{-\alpha} \\ \alpha(1-\alpha)x_1^{\alpha-1}x_2^{-\alpha} & (1-\alpha)(-\alpha)x_1^{\alpha}x_2^{-\alpha-1} \end{bmatrix} \sim \begin{bmatrix} -1/x_1^2 & 1/(x_1x_2) \\ 1/(x_1x_2) & -1/x_2^2 \end{bmatrix} = -\begin{bmatrix} 1/x_1 \\ -1/x_2 \end{bmatrix} \begin{bmatrix} 1/x_1 \\ -1/x_2 \end{bmatrix}^2$$

so it is negative definite. Hence f is concave and quasiconcave but not the other two.

Solution to 3.29. (Did he miss the word "convex" in the question?) Let $x \in \mathbb{R}^n$ and assume $x \in X_i$. Choose any $j \neq i$ and $y \in X_j$. If $y \notin \partial X_j$, pick $1 > \lambda > 0$ sufficiently small such that $y + \lambda(x-y) \in X_j$. By convexity,

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &= f(y + \lambda(x-y)) \\ &\leqslant \lambda f(x) + (1-\lambda)f(y) = f(y) + \lambda(f(x) - f(y)), \end{aligned}$$

so

$$\lambda f(x) \geqslant f(\lambda x + (1-\lambda)y) + \lambda f(y) - f(y)$$

and

$$f(x) \geqslant f(y) + \frac{f(y + \lambda(x-y)) - f(y)}{\lambda}.$$

Substituting this using the linear functions,

$$\begin{aligned} a_i^T x + b_i &\geqslant a_j^T y + b_j + \lambda^{-1}(a_j^T(y + \lambda(x-y)) + b_j - a_j^T y - b_j) \\ &= a_j^T y + b_j + \lambda^{-1}(a_j^T \lambda(x-y)) = a_j^T y + b_j + a_j^T(x-y) = a_j^T x + b_j. \end{aligned}$$

Same thing holds for $y \in \partial X_j$, in which case just let $\lambda = 0$. Taking maximum over all j we obtain the result.

Solution to 3.30. This follows from the fact that the convex hull of a set is the intersection of all convex supsersets of that set. Translated to the language of functions, the convex hull of the function is the pointwise supremum of all convex underestimators of f. This proves the claim.

Solution to 3.32. (1) Let x, y be given and let $\lambda \in (0,1)$. Then

$$\begin{aligned} f(\lambda x + (1-\lambda)y)g(\lambda x + (1-\lambda)y) &\leqslant (\lambda f(x) + (1-\lambda)f(y))(\lambda g(x) + (1-\lambda)g(y)) \\ &= \lambda^2 f(x)g(x) + (1-\lambda)^2 f(y)g(y) + \lambda(1-\lambda)(f(x)g(y) + f(y)g(x)) \\ &= [\lambda - \lambda(1-\lambda)]f(x)g(x) + [(1-\lambda) - \lambda(1-\lambda)]f(y)g(y) + \dots \\ &= \lambda f(x)g(x) + (1-\lambda)f(y)g(y) \\ &\quad + \lambda(1-\lambda)(-f(x)g(x) + f(x)g(y) + f(y)g(x) - f(y)g(y)) \\ &= \lambda f(x)g(x) + (1-\lambda)f(y)g(y) + \lambda(1-\lambda)(f(y) - f(x))(g(x) - g(y)). \end{aligned}$$

If f, g are both nonincreasing or both nondecreasing then $(f(y) - f(x))(g(x) - g(y)) \leqslant 0$. This proves the claim.

(2) Replace $\leqslant$ in (1) by $\geqslant$. The last term this time is positive, and the claim follows again.

(3) If g is concave, positive, nonincreasing, then $1/g$ is convex, positive, nondecreasing. The result follows from (1).