Problem 4.2

Proof. (1) If there exists v with $Av \leq 0$ then certainly $\lambda Av \leq 0$ for all $\lambda > 0$. This implies $x + \lambda v \in \text{dom}(f)$ for all $\lambda > 0$ and $x \in \text{dom}(f)$, making the domain unbounded.

Conversely, suppose the domain is unbounded and let $\{x^{(n)}\}$ be a sequence with $\|x^{(n)}\|_2 \to \infty$. Since ∂B_1 is compact, the sequence $\{x^{(n)}/\|x^{(n)}\|_2\}$ has a convergent subsequence. The limit, which we call x, satisfies $\|x\|_2 = 1$ and $Ax \leq 0$. To see this, recall that for each n we have $Ax^{(n)} \leq b$, so

$$A(x^{(n)}/\|x^{(n)}\|_2) \leq b/\|x^{(n)}\|_2.$$

Passing the inequality to the aforementioned subsequence and letting the index tend to ∞, we have $Ax \leq 0$.

(2) If such v exists, then for $\lambda > 0$ and $x \in \text{dom}(f)$,

$$f_0(x + \lambda v) = -\sum_{i=1}^{m} \log(b_i - a_i^T x - \lambda a_i^T v).$$

As λ increases, the terms $b_i - a_i^T x - \lambda a_i^T v$ increases. Letting $\lambda \to \infty$, the terms in the RHS $\to -\infty$, so f_0 is unbounded from below.

Conversely, suppose there exists a sequence $\{x^{(n)}\}$ with $(b - Ax^{(n)} \geq 0$, which we implicitly assume, and) $f_0(x^{(n)}) \to -\infty$. In particular, for some $j \in [1, m]$ we must have

$$\log(b_j - a_j^T x^{(n)}) \to \infty \implies \lim_{n\to\infty} \max_{1\leqslant j\leqslant m} (b_j - a_j^T x^{(n)}) = \infty.$$

Using the hint, suppose there exists z with $z > 0$ and $A^T z = 0$. Then $(A^T z)^T x^n = 0$. Therefore

$$z^T(b - Ax^{(n)}) = z^T b - z^T Ax^{(n)} = z^T b$$

whereas

$$z^T b = \sum_{i=1}^{m} z_i b_i \geqslant z_j b_j \to \infty.$$

(Here we abuse the notation, assuming that for a fixed m, $j := \text{argmax}_j(b_j - a_j^T x^{(n)})$.) Contradiction,

(3) If the domain is bounded then the sublevel sets are closed and therefore compact. Pick any sublevel set, and f_0 attains a minimum on it, and this minimum must also be the global minimum of f_0.

If the domain is unbounded, it needs to be unbounded in some direction. Let v be any vector such that, for all $M > 0$, there exists a scalar multiple of v with norm $> M$. (That is, v is a "direction" along which $\text{dom}(f)$ is unbounded.) It follows that $Av \leq 0$. By the previous part we must have $Av = 0$, so $f_0(\lambda v)$ is constant for all $\lambda > 0$. Excluding all such directions, we obtain a bounded set, so f_0 obtains a minimum on the remaining subset of $\text{dom}(f)$. Therefore the minimum must be attained in either case.

(4) f is strictly convex so there can be at most one optimal point. □

Problem 4.8

(1) If infeasible, the answer is ∞. Otherwise decompose c as $A^T v + w$ where w is in the nullspace of A. Then

$$c^T x = v^T Ax + w^T x = v^T b + w^T x.$$

If $w = 0$ then the optimal value is simply $v^T b$. Otherwise, any solutions of form $x + \lambda w$ works, so the answer is $-\infty$.

(2) $a^T x \leqslant b$ always has a solution so the system is feasible. We decompose c according to a: $c = \lambda a + d$ where $\lambda \in \mathbb{R}$ and $d^T a = 0$. Then

$$c^T x = (\lambda a + d)^T x = \lambda a^T x + d^T x.$$

If $\lambda > 0$ then the problem is unbounded from below by considering $x = -ta$, $t \to \infty$, since $a^T x = -ta^T a \to -\infty$ and $c^T x = -ta^T a \to -\infty$.

If $\lambda = 0$ then d is perpendicular to a. (If $c = 0$ there is nothing to show.) By considering $x = ba - tc$, we have

$$c^T x = bd^T a - tc^T c = -tc^T c \to -\infty$$

and indeed

$$a^T(ba - tc) = ba^T a - ta^T c < b \qquad \text{eventually.}$$

If $\lambda < 0$ and $d = 0$ then $c = \lambda a$, so $c^T x = \lambda a^T x \geqslant \lambda b$.

Finally, if $\lambda < 0$ and $d \neq 0$ then using $x = ba - tc$ we have $c^T x \to \infty$ once more as $t \to \infty$. Therefore,

$$p* = \begin{cases} ca/b & \text{if } c/a \in \mathbb{R}_- \\ -\infty & \text{otherwise.} \end{cases}$$

(3) It suffices to minimize componentwise:

$$x_i^* := \begin{cases} l_i & c_i > 0 \\ u_i & c_i \leqslant 0. \end{cases}$$

(4) This is a weighted average problem with minimum attained when all x_i are 0 except the one corresponding to the smallest component of c. In this case $c^T x$ is exactly the value of that component.

If $1^T x \leqslant 1$ instead, we set all $x_i = 0$ if the smallest component of c is positive, or we keep the answer in the $1^T x = 1$ case if the smallest component is negative.

(5) Similar to the previous part, if α is an integer then the minimum value corresponds to the sum of the α smallest components of c. If α is not an integer, the minimum is the sum of the $\lfloor \alpha \rfloor$ smallest compoenents of c, plus $(\alpha - \lfloor \alpha \rfloor)$ times the remaining smallest component.

If $1^T x \leqslant \alpha$ then we simply further require the chosen components to be nonpositive and replace the positive ones by 0.

(6) We instead consider

$$\begin{aligned} &\text{minimize } \sum_{i=1}^n (c_i/d_i) y_i \\ &\text{subject to } 1^T x = \alpha, 0 \leq y \leq d. \end{aligned}$$

The result then follows from the previous part.

Problem 4.9

Proof. Assuming A is invertible, minimizing $c^T x$ is equivalent to minimizing $c^T A^{-1} y = (A^{-T} c)^T y$ with constraint $y \leq b$. If $A^{-T} c \leq 0$ the optimal point is $y = b$ or $x = A^{-1} b$, so $p^* = c^T A^{-1} b$. Otherwise, letting $y \leq 0$ and $t \to \infty$, we have $(A^{-T} c)(ty) \to -\infty$. □

Problem 4.21

(1) Define $v := A^{1/2} x$ so that $x^T A x \leqslant 1$ becomes $\|v\|^2 \leqslant 1$. Then we are trying to minimize $c^T (A^{1/2})^T v$. Define $u^T := c^T (A^{1/2})^T$. The minimizer is $-u/\|u\|_2$. Translating this back to original variables,

$$v^* = -\frac{u}{\|u\|_2} = -\frac{A^{-1/2} c}{\|A^{-1/2} c\|} \implies x^* = A^{-1/2} v = -\frac{A^{-1} c}{\|A^{-1/2} c\|} = -\frac{A^{-1} c}{\sqrt{c^T A^{-1} c}}.$$

If A is not PD, diagonalize it as $A = QDQ^T$. Then $x^T A x \leqslant 1$ becomes $x^T Q D Q^T x \leqslant 1$, or $\|D^{1/2} Q x\| \leqslant 1$, and our objective function is $c^T x = c^T Q^T Q x = (Qc)^T (Qx)$. If all eigenvalues of A are positive then this is identical to the previous case.

If the smallest eigenvalue is negative, the answer is $-\infty$. If the smallest eigenvalue is 0, but the corresponding component of Qc is nonzero, then the answer is again unbounded from below. If all components of Qc corresponding to an eigenvalue of 0 are zero, then we reduce the problem into a smaller case with positive eigehvalues, and the result follows from the first case as well.

(2) This is identical to the first problem after a change of variable, yielding

$$x^* = x_c - \frac{A^{-1} c}{\sqrt{c^T A^{-1} c}}.$$

(3) If $B \geq 0$ then the minimum is 0 with $x = 0$. Otherwise, we define $y := A^{1/2} x$ and $C := (A^{-1/2})^T B A^{-1/2}$. Then our objective function becomes $y^T C y$ and the constraint becomes $\|y\| \leqslant 1$. Therefore, the optimal value is given by

$$p^* = \min \left\{ 0, \min_{\|x\|=1} x^T (A^{-1/2})^T B A^{-1/2} x \right\}.$$

Since any arbitrary vector is a linear combination of eigenvalues, the second quantity is uniquely minimized when y is the smallest eigenvalue of C.

Problem 4.22

Proof. The gradient of the quadratic function is $Px + q$. Given that the objective is convex, we either have a unique minimum in the interior, at which the gradient vanishes, or we attain minimum on the boundary.

Therefore, if $Px + q = 0$ has a solution with $\|x\| \leqslant 1$, we are done. Otherwise, we want to use Lagrange multipliers and solve

$$\|x\| = 1 \qquad \text{and} \qquad Px + q = -\lambda x, \lambda \in \mathbb{R}.$$

Therefore $(P + \lambda I)x = -q$ for some $\lambda > 0$, giving the solution $x = -(P + \lambda I)^{-1} q$. From this we recover the claim. □