MATH 520 Homework 1

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Problem 1

Prove directly from definition that e^z is differentiable at 0. Also prove directly that the definition that z^n is differentiable and $(z^n)' = nz^{n-1}$.

Proof. (1) *I assume I need to use power series representation* $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$. Let $f(z) := e^z$. Then

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{f(h)-1}{h} \\ &= \lim_{h\to 0} \left(\frac{1}{h} + \sum_{n=1}^{\infty} \frac{h^n}{n!\cdot h} - \frac{1}{h} \right) \\ &= \lim_{h\to 0} \sum_{n=1}^{\infty} \frac{h^{n-1}}{n!} = \lim_{h\to 0} \left(1 + \frac{h}{2!} + \frac{h^2}{3!} + \cdots \right). \end{aligned}$$

For convenience define the "error" $\text{err}(h) := h/2! + h^2/3! + \ldots$ so that

$$\frac{f(h)-1}{h} = 1 + \text{err}(h).$$

We then have

$$\left| \frac{f(h)-1}{h} - 1 \right| = |\text{err}(h)| \leqslant \sum_{n=2}^{\infty} \frac{|h|^{n-1}}{n!}.$$

Let $\epsilon > 0$. If we pick $\delta = \min(\epsilon, 1/2)$ and if $|h| < \delta$, then

$$|\text{err}(h)| < \sum_{n=2}^{\infty} \frac{\delta^{n-1}}{n!} < \sum_{n=2}^{\infty} \frac{\delta}{n!} < \frac{\delta}{2!} + \frac{\delta}{3!} + \sum_{n=4}^{\infty} \frac{\delta}{2^n} = \frac{\delta}{2} + \frac{\delta}{6} + \frac{\delta}{8} = \frac{19\delta}{24} < \epsilon.$$

This proves that $f'(0) = \lim((f(h)-1)/h) = 1$, as anyone who has taken Calculus I would expect.

(2) The proof of the second claim is identical to its real-valued counterpart:

$$\begin{aligned} (z^n)' &= \lim_{h\to 0} \frac{(z+h)^n - z^n}{h} = \lim_{h\to 0} \frac{1}{h} \left(\sum_{k=1}^{n} \binom{n}{k} z^{n-k} h^k \right) \\ &= \sum_{k=1}^{n} \lim_{h\to 0} \binom{n}{k} z^{n-k} h^{k-1} = nz^{n-1} + \sum_{k=2}^{n} 0 = nz^{n-1}. \end{aligned}$$

□

Problem 2: (Alhfors, p.28 problem 2, modified)

Verify Cauchy-Riemann's equations for the functions z^2, z^3, e^z, and the principle branch of $\log z$.

Solution. For convenience we define $x = \mathfrak{Re} z$ and $y = \mathfrak{Im} z$ so that $z = x + iy$.

(1) $z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy$. Now we verify the Cauchy-Riemann conditions: indeed,

$$\frac{\partial}{\partial x}(x^2 - y^2) = 2x = \frac{\partial}{\partial y}(2xy) \qquad \text{and} \qquad \frac{\partial}{\partial y}(x^2 - y^2) = -2y = -\frac{\partial}{\partial x}(2xy).$$

(2) $z^3 = (x+iy)^3 = x^3 + 3ix^2y - 3xy^2 - iy^3 = (x^3 - 3xy^2) + i(3x^2y - y^3)$. Indeed,

$$\frac{\partial}{\partial x}(x^3 - 3xy^2) = 3x^2 - 3y^2 = \frac{\partial}{\partial y}(3x^2y - y^3) \qquad \text{and} \qquad \frac{\partial}{\partial y}(x^3 - 3xy^2) = -6xy = -\frac{\partial}{\partial x}(3x^2y - y^3).$$

(3) We write $e^z = e^{x+iy} = e^x(\cos y + i \sin y) = e^x \cos y + ie^x \sin y$. Then,

$$\frac{\partial}{\partial x}(e^x \cos y) = e^x \cos y = \frac{\partial}{\partial y}(e^x \sin y),$$

and

$$\frac{\partial}{\partial y}(e^x \cos y) = -e^x \sin y = -\frac{\partial}{\partial x}(e^x \sin y).$$

(4) The principle branch is given by $f(z) = \log|z| + i \arg z$ so $f = u + iv$ where

$$u = \log\sqrt{x^2 + y^2} \qquad \text{and} \qquad v = \tan^{-1}(y/x).$$

Checking the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2} = \frac{\partial v}{\partial y} \qquad \text{and} \; \frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2} = -\frac{\partial v}{\partial x}.$$

Problem 3: (Alhfors, p.28 problem 5)

Prove that the functions $f(z)$ and $\overline{f(\overline{z})}$ are simultaneously analytic (i.e., both are or both aren't).

Proof. We will show that either both $f(z)$ and $\overline{f(\overline{z})}$ satisfy the Cauchy-Riemann equations or both fail to. As usual, let $x = \mathfrak{Re} z$, $y = \mathfrak{Im} z$, $u = \mathfrak{Re} f$, and $v = \mathfrak{Im} f$. Then

$$f(z) = u(x,y) + iv(x,y),$$

and

$$f(\overline{z}) = f(x - iy) = u(x,-y) + iv(x,-y)$$

so

$$\overline{f(\overline{z})} = u(x,-y) - iv(x,-y).$$

Let $\tilde{u}$ and $\tilde{v}$ denote the real and imaginary parts of $\overline{f(\overline{\cdot})}$; we have

$$\frac{\partial \tilde{u}}{\partial x}(x,y) = \frac{\partial u}{\partial x}(x,-y) \qquad \frac{\partial \tilde{v}}{\partial y}(x,y) = (-1)^2 \frac{\partial v}{\partial y}(x,-y) = \frac{\partial v}{\partial y}(x,-y) \tag{3.1}$$

and

$$\frac{\partial \tilde{u}}{\partial y}(x,y) = -\frac{\partial u}{\partial y}(x,-y) \qquad \frac{\partial \tilde{v}}{\partial x} = -\frac{\partial v}{\partial x}(x,-y). \tag{3.2}$$

It is clear from (3.1) and (3.2) that if one between $f(\cdot)$ and $\overline{f(\overline{\cdot})}$ is analytic then the other one must also be and vice versa. □

Problem 4: (Alhfors, p.28 problem 7)

Prove that a harmonic function satisfies the formal differential equation $\dfrac{\partial^2 u}{\partial z \partial \overline{z}} = 0$.

Proof. Using the definition $\overline{z} = x - iy$, along with identities $\dfrac{\partial u}{\partial z} = \dfrac{1}{2}\left(\dfrac{\partial u}{\partial x} - \dfrac{\partial u}{\partial y}\right)$ and $\dfrac{\partial u}{\partial \overline{z}} = \dfrac{1}{2}\left(\dfrac{\partial u}{\partial x} + i\dfrac{\partial u}{\partial y}\right)$, we have

$$\begin{aligned}\frac{\partial^2 u}{\partial z \partial \overline{z}} &= \frac{1}{4}\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right) \\ &= \frac{1}{4}\left(\frac{\partial^2 u}{\partial x^2} + i\frac{\partial^2 u}{\partial x \partial y} - i\frac{\partial^2 u}{\partial y \partial x} + \frac{\partial^2 u}{\partial y^2}\right) = 0.\end{aligned}$$ □

Problem 5: (Alhfors, p.72 problem 1)

Give a precise definition of a single-valued branch of $\sqrt{1+z} + \sqrt{1-z}$.

Solution. We first claim that $\sqrt{z}$ is single-valued on $\mathbb{C}\backslash(-\infty, 0]$ by the mapping $z = re^{i\theta} \mapsto \sqrt{r}e^{i\theta/2}$. Therefore, $\sqrt{1+z}$ is single-valued on $\mathbb{C}\backslash(-\infty,-1]$ and $\sqrt{1-z}$ is single-valued on $\mathbb{C}\backslash[1,\infty)$. Therefore, removing both branches cuts, we see $\sqrt{1+z} + \sqrt{1-z}$ is single-valued on $(\mathbb{C}\backslash\mathbb{R}) \cup (-1,1)$, indeed an open, connected set.

Problem 6

Prove that $f(z) = \overline{z}$ is not differentiable at any point in $\mathbb{C}$.

Proof. Let $z = a + ib \in \mathbb{C}$ be given. We show that the derivative does not exist by taking approaching $a + bi$ horizontally and vertically and showing the quotient limits do not agree:

$$\lim_{\substack{h \searrow 0 \\ h \in \mathbb{R}}} \frac{f(z+h) - f(z)}{h} = \lim \frac{a + h - ib - a + bi}{h} = 1$$

whereas

$$\lim_{\substack{h \searrow 0 \\ h \in \mathbb{R}}} \frac{f(z+ih) - f(z)}{ih} = \lim \frac{a - ih - ib - a + bi}{ih} = -1.$$ □

Problem 7: (Conway, p.43 problem 1)

Prove that $f(z) = |z|^2$ has a derivative only at the origin.

Proof. Using a similar approach as above, if $z = a + ib$, $a, b \in \mathbb{R}$, then

$$\lim_{\substack{h \searrow 0 \\ h \in \mathbb{R}}} \frac{|z+h|^2 - |z|^2}{h} = \lim \frac{|a+ib+h|^2 - |a+ib|^2}{h} = \lim \frac{(a+h)^2 + b^2 - a^2 - b^2}{h} = \lim \frac{2ah + h^2}{h} = 2a,$$

and

$$\lim_{\substack{h \searrow 0 \\ h \in \mathbb{R}}} \frac{|z+ih|^2 - |z|^2}{ih} = \lim \frac{|a+ib+ih|^2 - |a+ib|^2}{ih} = \lim \frac{a^2 + (b+h)^2 - a^2 - b^2}{ih} = \lim \frac{2bh + h^2}{ih} = -2bi.$$

If f is differentiable at $z = a + bi$ then it must be that $2a = -2bi$. Since $a, b \in \mathbb{R}$ this can only happen at the origin. Now we verify that f is indeed differentiable at origin:

$$\lim_{(a,b)\to(0,0)} \frac{f(0 + (a+ib)) - f(0)}{a+ib} = \lim_{(a,b)\to(0,0)} \frac{|a+ib|^2}{a+ib} = \lim_{(a,b)\to(0,0)} (a - ib) = 0. \qquad \square$$