MATH 520 Homework 10

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Problem 1: Alhfors, page 154 p1

How many roots does the equation $z^7 - 2z^5 + 6z^3 - z + 1 = 0$ have in the disk $|z| < 1$?

Solution. Let $g(z) :=$ the polynomial described above and let $f(z) := 6z^3$. Let γ be the boundary of the unit disk. Then, on γ,

$$|f(z) - g(z)| = |-z^7 + 2z^5 + z - 1| \leqslant |z^7| + 2|z^5| + |z| + 1 \leqslant 6 = |f(z)|.$$

Applying Rouche's theorem we see that g must have three roots in the disk since $6z^3$ (only) has a root of multiplicity of 3 at the origin.

Problem 2: Alhfors, page 154 p3

How many roots of the equation $z^4 + 8z^3 + 3z^2 + 8z + 3 = 0$ lie in the right half plane?

Solution. For points of form it, $t \in \mathbb{R}$ on the imaginary axis, we have

$$f(it) = z^4 - 8it^3 - 3t^2 + 8it + 3 = (z^4 - 3t^2 + 3) + i(-8t^3 - 8t).$$

It follows that $\mathfrak{Re} f(it) > 0$ for all t so the imaginary axis is always mapped to the right half plane. As $t \to \infty$, $\mathfrak{Re} f(it)$ dominates $\mathfrak{Im} f(it)$, so the change in argument of $f(it)$ eventually $\to 0$ as $t \to \infty$. Now consider the right semicircle with radius R and center origin. The degree of f is 4, so on $\theta \in (-\pi/2, \pi/2)$, for large R, the argument of $f(Re^{i\theta})$ wraps around the origin twice. Therefore the polynomial has two roots in the right half plane.

Problem 3: Alhfors, page 161 p3a

Compute

$$\int_0^{\pi/2} \frac{1}{a + \sin^2 x} \, \mathrm{d}x.$$

Solution. Using $\cos(2x) = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x$ we have

$$\begin{aligned}\int_0^{\pi/2} \frac{1}{a+\sin^2 x}\,\mathrm{d}x &= \int_0^{\pi/2} \frac{1}{a+(1-\cos(2x))/2}\,\mathrm{d}x \\ &= 2\int_0^{\pi/2} \frac{1}{2a+1-\cos(2x)}\,\mathrm{d}x \\ &= \int_0^{\pi} \frac{1}{(2a+1)-\cos x}\,\mathrm{d}x = \int_{-\pi}^{0} \frac{1}{(2a+1)+\cos x}\,\mathrm{d}x = \int_0^{\pi} \frac{1}{(2a+1)+\cos x}\,\mathrm{d}x.\end{aligned}$$

Using example 1 on Alhfors page 155, we have

$$\int_0^{\pi} \frac{1}{(2a+1)+\cos x}\,\mathrm{d}x = \pi/\sqrt{4a^2+4a}.$$

Problem 4: Alhfors, page 161 p3b

Compute

$$\int_0^{\infty} \frac{x^2}{x^4+5x^2+6}\,\mathrm{d}x.$$

Solution. Note that

$$z^4+5z^2+6 = (z^2+3)(z^2+2)(z+i\sqrt{3})(z-i\sqrt{3})(z+i\sqrt{2})(z-i\sqrt{2}).$$

We use upper semicircles as contours so that the poles contained are $i\sqrt{2}$ and $i\sqrt{3}$. Their residues are

$$\text{Res}(i\sqrt{2}) = \frac{(i\sqrt{2})^2}{(2i\sqrt{2})((i\sqrt{2})^2+3)} = \frac{-2}{i2\sqrt{2}} = \frac{i}{\sqrt{2}}$$

and

$$\text{Res}(i\sqrt{3}) = \frac{(i\sqrt{3})^2}{(2i\sqrt{3})((i\sqrt{3})^2+2)} = \frac{-3}{-i2\sqrt{3}} = \frac{-i\sqrt{3}}{2}.$$

Given R, the radius of the semicircle, the integral along the curve, which we denote as γ_0, is bounded by

$$\left|\int_{\gamma_0} \frac{x^2}{x^4+5x^2+6}\,\mathrm{d}x\right| \leqslant \pi R \cdot \frac{R^2}{|R^4-5R^2-6|}$$

whose limit is 0 as $R\to\infty$. Therefore the residue theorem gives

$$\int_{-\infty}^{\infty} \frac{x^2}{x^4+5x^2+6}\,\mathrm{d}x = 2\pi i \sum \text{Res} = \pi(\sqrt{3}-\sqrt{2})$$

and since the function is even,

$$\int_0^{\infty} \frac{x^2}{x^4+5x^2+6}\,\mathrm{d}x = \frac{\pi(\sqrt{3}-\sqrt{2})}{2}.$$

Problem 5: Alhfors, page 161 p3e

Compute

$$\int_0^{\infty} \frac{\cos x}{x^2+a^2}\,\mathrm{d}x.$$

Solution. Let γ be the curve connecting $(-r,0),(0,r)$, and the upper semicircle from $(r,0)$ to $(-r,0)$, oriented counterclockwise. Assume $r > a$. Since $\exp(ix)/(x^2+a^2)$ only has zero at $x = ia$, we see that it is included in the region enclosed by γ. Hence

$$\begin{aligned}\int_0^\infty \frac{\cos(x)}{x^2+a^2}\,\mathrm{d}x &= \frac{1}{2}\int_{-\infty}^\infty \frac{\cos(x)}{x^2+a^2}\,\mathrm{d}x \\ &= \frac{1}{2}\mathfrak{Re}\int_{-\infty}^\infty \frac{\exp(ix)}{x^2+a^2}\,\mathrm{d}x \\ &= \frac{1}{2}\mathfrak{Re}\int_\gamma \frac{\exp(ix)}{x^2+a^2}\,\mathrm{d}x \\ &= \frac{1}{2}\mathfrak{Re}(2\pi i \mathrm{Res}(ia)) = \frac{1}{2}\mathfrak{Re}\left(2\pi i \lim_{z\to ia}(z-ia)\frac{\exp(iz)}{z^2+a^2}\right) \\ &= \frac{1}{2}\mathfrak{Re}\left(2\pi i \lim_{z\to ia}\frac{\exp(iz)}{z+ia}\right) = \frac{1}{2}\mathfrak{Re}(2\pi i\exp(-a)/(2ia)) = \frac{\pi\exp(-a)}{2a}.\end{aligned}$$

(It is almost trivial that the upper semicircle has integral $\to 0$ as $R\to\infty$, since the integral is bounded by $\pi R\cdot 1/R^2 \to 0$.)

Problem 6: Alhfors, page 161 p3g

Compute

$$\int_0^\infty \frac{x^{1/3}}{1+x^2}\,\mathrm{d}x.$$

Solution. We use Alhfors's method in the text. Transforming the integral of form $\int_0^\infty x^\alpha R(x)\,\mathrm{d}x$ into form $2\int_0^\infty t^{2\alpha+1}R(t^2)\,\mathrm{d}t$,

$$\int_0^\infty \frac{x^{1/3}}{1+x^2}\,\mathrm{d}x = 2\int_0^\infty \frac{x^{5/3}}{1+x^4}\,\mathrm{d}x$$

and

$$\int_{-\infty}^\infty \frac{x^{5/3}}{1+x^4}\,\mathrm{d}x = \int_0^\infty \frac{z^{5/3}+(-z)^{5/3}}{1+z^4}\,\mathrm{d}z = (1-\exp(2\pi i/3))\int_0^\infty \frac{z^{5/3}}{1+z^2}\,\mathrm{d}z,$$

so

$$\int_0^\infty \frac{x^{1/3}}{1+x^2}\,\mathrm{d}x = \frac{2}{1-e^{2\pi i/3}}\int_{-\infty}^\infty \frac{z^{5/3}}{1+z^4}\,\mathrm{d}z.$$

Also consider the upper half of an $R-\epsilon$ annulus, along with two line segments of real axis, as shown in Alhfors's figure 4.13 (a large semicircle with an ϵ-semicircle removed). The function $z^{5/3}/(1+z^4)$ has two singularities in the upper plane, namely $e^{\pi i/4}$ and $e^{3\pi i/4}$.

Note that

$$z^4+1 = (z-e^{\pi i/4})(z-e^{3\pi i/4})(z-e^{5\pi i/4})(z-e^{7\pi i/4}),$$

so the residues are

$$\mathrm{Res}(e^{\pi i/4}) = \frac{e^{5\pi i/12}}{(e^{\pi i/4}-e^{3\pi i/4})(e^{\pi i/4}-e^{5\pi i/4})(e^{\pi i/4}-e^{7\pi i/4})} = \frac{1}{4}e^{-\pi i/3}$$

and

$$\mathrm{Res}(e^{3\pi i/4}) = \frac{e^{15\pi i/12}}{(e^{3\pi i/4}-e^{\pi i/4})(e^{3\pi i/4}-e^{5\pi i/4})(e^{3\pi i/4}-e^{7\pi i/4})} = -\frac{1}{4}.$$

It is clear that as $R \to \infty$ and $\epsilon \to 0$, both semicircles have integrals whose values tend to 0, since the corresponding integrals are of order $1 + 1/3 - 2 < 0$. Therefore, using residue theorem, we obtain

$$\int_0^\infty \frac{x^{1/3}}{1+x^2}\,\mathrm{d}x = \frac{2}{1-e^{2\pi i/3}} 2\pi i(\mathrm{Res}(e^{\pi i/4}) + \mathrm{Res}(e^{3\pi i/4})) = \frac{\pi}{\sqrt{3}}.$$

Problem 7: Alhfors, page 161 p3h

Compute

$$\int_0^\infty \frac{\log x}{1+x^2}\,\mathrm{d}x.$$

Solution. We again use the $R-\epsilon$ annulus as contour. The function $\log x/(1+x^2)$ has one zero enclosed by such annulus at $z = i$. The residue at $z = i$ is

$$\mathrm{Res}(i) = \frac{\log(i)}{i+i} = \frac{\log(e^0 e^{i\pi/2})}{2i} = \frac{i\pi/2}{2i} = \frac{\pi}{4}.$$

As $\epsilon \to 0$, the integral along the ϵ-semicircle is bounded by

$$\left| \int_{\gamma_\epsilon} \frac{\log x}{1+x^2}\,\mathrm{d}x \right| \leqslant \pi\epsilon \sup_{\gamma_\epsilon} \frac{|\log z|}{|1+z^2|} \to 0$$

and clearly the other bound for R holds, as R^2 dominates $R\log R$. It remains to notice that

$$\begin{aligned}
\int_{-\infty}^\infty \frac{\log x}{1+x^2}\,\mathrm{d}x &= \int_{-\infty}^0 \frac{\log x}{1+x^2}\,\mathrm{d}x + \int_0^\infty \frac{\log x}{1+x^2}\,\mathrm{d}x \\
&= \int_0^\infty \frac{\log x}{1+x^2}\,\mathrm{d}x + \int_{-\infty}^0 \frac{\log(-x)+\pi i}{1+(-x)^2}\,\mathrm{d}x \\
&= 2\int_0^\infty \frac{\log x}{1+x^2}\,\mathrm{d}x + \pi i \int_{-\infty}^0 \frac{1}{1+x^2}\,\mathrm{d}x \\
&= 2\int_0^\infty \frac{\log x}{1+x^2}\,\mathrm{d}x + \pi i(\pi/2).
\end{aligned}$$

This, along with residue theorem which gives

$$\int_{-\infty}^\infty \frac{\log x}{1+x^2}\,\mathrm{d}x = 2\pi i(\pi/4) = \frac{\pi^2 i}{2}$$

implies

$$\int_0^\infty \frac{\log x}{1+x^2}\,\mathrm{d}x = 0.$$