MATH 520 Homework 11

Qilin Ye

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Problem 1: Alhfors, 186.1

Prove that Laurent development is unique.

Proof. Suppose f is analytic in the annulus $r < |z - z_0| < R$ and

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n = \sum_{n=-\infty}^{\infty} b_n (z - z_0)^n.$$

Then, for any integer k,

$$f(z)(z - z_0)^{-k-1} = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^{n-k-1} = \sum_{n=\infty}^{\infty} b_n (z - z_0)^{n-k-1}.$$

Let γ be any closed curve in the annulus so it is in particular compact. Hence uniform convergence implies that ntegral and summation commute, and so

$$\sum_{n=-\infty}^{\infty} a_n \int_\gamma (z - z_0)^{n-k-1}\,\mathrm{d}z = \sum_{n=-\infty}^{\infty} b_n \int_\gamma (z - z_0)^{n-k-1}\,\mathrm{d}z.$$

It remains to notice that for k, all integrals evaluate to 0 except when $n = k$, in which case we have $2\pi i a_n = a\pi i b_n$, i.e., $a_n = b_n$. Letting k vary we obtain the result. □

Problem 2: Alhfors 186.3

The expression

$$\{f, z\} = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$$

is called the Schwarzian derivative of f. If f has a multiple zero or pole, find the leading term in the Laurent development of $\{f, z\}$.

Solution. First note that

$$\frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 = \frac{f'''(z)f'(z)}{(f'(z))^2} - \left(\frac{f''(z)}{f'(z)}\right)^2 - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2.$$

If f has a multiple zero or pole, then the leading term $f(z) = a(z - z_0)^m + \ldots$ have $|z| \geqslant 2$. Then $f'(z) = am(z - z_0)^{m-1} + \ldots$ and so $f''(z) = am(m-1)(z - z_0)^{m-2} + \ldots$. Therefore

$$\frac{f''(z)}{f'(z)} = \frac{m-1}{z - z_0} + \mathcal{O}(1).$$

This implies

$$\left(\frac{f''(z)}{f'(z)}\right)' = -\frac{m-1}{(z-z_0)^2} + \dots \qquad \text{and} \qquad \left(\frac{f''(z)}{f'(z)}\right)^2 = \frac{(m-1)^2}{(z-z_0)^2} + \dots$$

so $\{f, z\} = (1-m^2)/2 \cdot (z-z_0)^{-2} + \dots$.

Problem 3: Alhfors, 193.1

Show that

$$\prod_{n=2}^{\infty}\left(1 - \frac{1}{n^2}\right) = \frac{1}{2}.$$

Proof. Writing $1 - 1/n^2$ as $(n-1)(n+1)/n^2$, the partial product $\prod_{n=2}^{k}\left(1-\frac{1}{n^2}\right) = \frac{1}{2}(1+1/k)$. Then take limit. □

Problem 4: Alhfors, 193.3

Prove that

$$\prod_{n=1}^{\infty}\left(1 + \frac{z}{n}\right)e^{-z/n}$$

converges absolutely and uniformly on every compact set.

Proof. Showing the claim is equivalent to showing the convergence of $\sum_{n=1}^{\infty}\left|\log\left(1+\frac{z}{n}\right) - \frac{z}{n}\right|$. Note that the function $(\log(1+z)-z)/z^2$ is locally analytic around $z=0$ and so is bonuded. Any compact set K is bounded, and for $z \in K$ and sufficiently large n_0 we have $|z/n_0| < 1/2$. Then

$$\sum_{n \geqslant n_0}\left|\log\left(1+\frac{z}{n}\right) - \frac{z}{n}\right| = \sum_{n\geqslant n_0}\left|(z/n)^2(\log(1+z/n) - z/n)/(z/n)^2\right| \leqslant C\sum_{n\geqslant n_0}\left(\frac{z}{n}\right)^2 < \infty.$$

for some C, for example $\sup_{z\in K}(\log(1+z)-z)/z^2$. The finitely many terms not included does not change the convergence of the series. □

Problem 5: Alhfors, 193.4

Prove that the value of an absolutely convergent product does not change if the factors are reordered.

Proof. By definition, $\prod_{n=1}^{\infty}(1+a_n)$ converges absolutely iff $\sum_{n=1}^{\infty}\log(1+a_n)$ converges absolutely, so it suffices to show that the latter series have constant sum under rearrangements. Suppose for contradiction that this is not the case. That any rearrangement still converges absolutely is trivial, for the partial sums form an increasing sequence bounded by the original infinite sum $a := \sum_{n=1}^{\infty}\log(1+a_n)$. For convenience we denote it as $\sum_{n=1}^{\infty} b_n$. Now suppose $\sum_{n=1}^{\infty} c_n$ is an rearrangement. For $\epsilon > 0$ there exists N such that $\left|\sum_{n=1}^{N} b_n - a\right| < \epsilon$ and $\sum_{n\geqslant N}|b_n| < \epsilon$. Let N' be large so that $c_1, \dots, c_{N'}$ include the first N terms of $\{a_n\}$. Then

$$\left|\sum_{n=1}^{N'} c_n - a\right| \leqslant \left|\sum_{n=1}^{N} a_n - a\right| + \sum_{k\geqslant N}|a_k| < 2\epsilon.$$

Since ϵ is arbitrary, we must have $\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} b_n$, completing our proof. □

Problem 6: Conway, 127,8

Is a non-constant meromorphic function on a region G an open mapping of G into $\mathbb{C}$? Is it an open mapping of G into $\mathbb{C}_\infty$?

Solution. No, consider G consisting of two connected components and a function f taking a different constant value on each one.

If we assume G is connected, clearly not every meromorphic has codomain $\mathbb{C}$. It is, however, open from G into $\mathbb{C}_\infty$. Let f be metamorphic. If $f(z) \neq \infty$ and $z \neq \infty$ we are done. If $f(z) = \infty$ and $z \neq \infty$, we consider $1/f(z)$, an analytic function. If $f(z) \neq \infty$ and $z = \infty$, we consider $f(1/z)$, also analytic. Finally, if $f(z) = z = \infty$, then we consider $1/f(1/z)$. All cases follow from the result that an analytic mapping is open.

Problem 7: Conway 127.9

Let $\lambda > 1$ and show that $\lambda - z - e^{-z} = 0$ has exactly one solution in the half plane $\{z : \mathfrak{Re} z > 0\}$. Show that this solution is real. What happens to the solution as $\lambda \to 1$?

Proof. If $\mathfrak{Re} z > 0$ and $\lambda - z - e^{-z} = 0$ then $|\lambda - z| = |e^{-z}| = e^{-\mathfrak{Re} z} < 1$.
Now by Rouche's theorem, since on the boundary $\partial D_1(z)$,

$$|(\lambda - z) - (\lambda - z - e^{-z})| = |e^{-z}| \leqslant |\lambda - z|,$$

the function $\lambda - z - e^{-z}$ and $\lambda - z$ have same number of roots in $D_1(z)$. This proves the uniqueness claim.
Also, since $\lambda - 0 - e^{-0} = \lambda - 1 < 0$ and $\lambda - \lambda - e^{\lambda} < 0$, IVT implies there is a root on the real axis. Combing the results we obtain our claim. □