MATH 520 Homework 2

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Problem 1: Alhfors, p.32 problem 2

If Q is a polynomial with distinct roots $\alpha_1, \ldots, \alpha_n$ and P a polynomial of degree $< n$, show that

$$\frac{P(z)}{Q(z)} = \sum_{k=1}^{n} \frac{P(\alpha_k)}{Q'(\alpha_k)(z - a_k)}.$$

Proof. WLOG assume Q is monic, i.e., it is of form $Q(z) = \prod_{i=1}^{n}(z - \alpha_i)$. A simple induction on n shows that

$$Q'(z) = \sum_{i=1}^{n} \prod_{j \neq i}(z - \alpha_j)$$

and in particular

$$Q'(\alpha_k) = \prod_{j \neq k}(z - \alpha_j).$$

Therefore,

$$Q(z) \sum_{k=1}^{n} \frac{P(\alpha_k)}{Q'(\alpha_k)(z - \alpha_k)} = \sum_{k=1}^{n} \frac{P(\alpha_k) \prod_{i=1}^{n}(z - \alpha_i)}{\prod_{j \neq k}(z - \alpha_j)(z - \alpha_k)} = \sum_{k=1}^{n} \frac{P(\alpha_k) \prod_{i \neq k}(z - a_i)}{\prod_{j \neq k}(z - \alpha_j)}. \tag{1}$$

If z takes the value of one of the roots of Q, then the summation has $n - 1$ terms of 0 and one term of $P(a_k)$, so $P(a_k)$ agrees with (1) at each $x = \alpha_k$. Given that P has degree $< n$, they must be uniformly equal. Dividing both sides by $Q(z)$ gives the claim. □

Problem 2: Alhfors, p.32 problem 3

Use the formula in the preceding exercise to prove that there exists a unique polynomial P of degree $< n$ with given values c_k at points α_k. (Lagrange's interpolation polynomial).

Proof. Already done. □

Problem 3

What is the general form of a rational function which maps $\mathbb{R}$ to $\mathbb{R}$? In particular, how are the zeros and poles related to each other? (Hint: if R is the rational function, consider the difference $R(z) - \overline{R(\overline{z})}$.)

Solution. If R maps $\mathbb{R}$ to $\mathbb{R}$, and if $r \in \mathbb{R}$, then $R(r) = R(\overline{r}) = \overline{R(\overline{r})}$. Therefore the function $R(z) - \overline{R(\overline{z})}$ equals 0 on the entire real line. Since this is also a rational function, the entire function must be the zero constant function. From this we also have $1/R(z) = 1/\overline{R(\overline{z})}$.

Therefore, if z is a zero of R, we must have $R(\overline{z}) = \overline{R(\overline{z})} = 0$, i.e., $\overline{z}$ is also a zero of R. Similarly, if z is a pole of R then it is a zero of $1/R(z)$, so it must be a zero of $1/\overline{R(\overline{z})}$, and so $\overline{z}$ must also be a pole of R. This implies that we have an even number of (not necessarily distinct) zeros and poles.

If we let $\{a_i\}_{i=1}^{2n}$ be the collection of zeros, not necessarily distinct, then we can partition them into $\{b_i\}_{i=1}^{n}, \{c_i\}_{i=1}^{n}$, where $b_i = \overline{c_i}$ for $1 \leqslant i \leqslant n$. Likewise if $\{p_j\}_{j=1}^{2m}$ are the poles then we can partition them into $\{r_j\}_{j=1}^{m}$ and $\{s_j\}_{j=1}^{m}$ with $r_j = \overline{s_j}$. By doing so, we obtain the general form

$$R(z) = \frac{\prod_{i=1}^{n}(z - b_i)(z - \overline{b_i})}{\prod_{j=1}^{m}(z - r_j)(z - \overline{r_j})}.$$

Problem 4: Alhfors, p.33 problem 4

What is the general form of a rational function which has absolute value 1 on the circle $|z| = 1$? In particular, how are the zeros and poles related to each other? (Hint: if R is the rational function, consider $R(z)\cdot\overline{R(1/\overline{z})}$).

Solution. If $|z| = 1$ then $z\overline{z} = 1$, i.e., $1/\overline{z} = z$, and the assumption implies $f(z) := R(z)\overline{R(1/\overline{z})} = 1$ whenever $|z| = 1$. Since $f \equiv 1$ on infinitely many points, f must be the constant function. That is, we have $R(z) = 1/\overline{R(1/\overline{z})}$.

If z_0 is a zero of R then $1/\overline{z_0}$ must be a pole and vice versa; furthermore, the orders must all agree.

Therefore, if $\{a_i\}_{i=1}^{n}$ are the zeros of R (if a root has order > 1, list it multiple times in the set), then $\{1/\overline{a_i}\}_{i=1}^{n}$ are the poles, and we have a candidate

$$g(z) := \prod_{i=1}^{n}\left(\frac{z - a_i}{z - 1/\overline{a_i}}\right).$$

Moreover, multiplying this function by any constant with modulus 1 does not ruin the desired property, so the general form is

$$f(z) = c\prod_{i=1}^{n}\left(\frac{z - a_i}{z - 1/\overline{a_i}}\right)$$

where $|c| = 1$ and $\{a_i\}_{i=1}^{n}$ are the zeroes of R, not necessarily distinct.

Problem 5: Alhfors, p.78 problem 1

Prove that the reflection $z \mapsto \overline{z}$ is not a fractional linear transformation.

Proof. Suppose for contradiction that $z \mapsto \overline{z}$ is fractional linear. That is, there exists $a, b, c, d \in \mathbb{C}$ such that

$$\frac{a(x + iy) + b}{c(x + iy) + d} = x - iy \qquad \text{for all } (x, y) \in \mathbb{R}^2.$$

Picking $x = 0, y = 1$ and $x = 0, y = -1$ give

$$\begin{cases} \dfrac{ai + b}{ci + d} = -i \\ \dfrac{-ai + b}{-ci + d} = i \end{cases} \implies \begin{cases} c - di = ai + b \\ c + di = -ai - b \end{cases} \implies \begin{cases} 2c = 2b \\ 2di = -2ai \end{cases} \implies \begin{cases} b = c \\ a = -d \end{cases}.$$

Then, picking $x = 1, y = 0$ gives

$$\frac{a+b}{c+d} = \frac{a+b}{b-a} = 1 \implies a = 0.$$

Therefore, for all $(x, y) \in \mathbb{R}^2$ we have

$$\frac{a(x+yi)+b}{c(x+yi)+d} = \frac{b}{b(x+yi)} = \frac{1}{x+yi} = x - yi,$$

which is absurd. Hence $z \mapsto \overline{z}$ is not fractional linear. □

Problem 6: Alhfors, p.78 problem 3

Prove that the most general transformation which leaves the origin fixed and preserves all distances is either a rotation or a rotation followed by reflection in the real axis.

Proof. Let f be such a function. By assumption $f(0) = 0$ and $|f(1)| = 1$. WLOG assume $f(1) = 1$ (for if $f(1) = e^{i\theta}$, we can first rotate by $-\theta$ and doing so does not break the claim).

Since two triangles are congruent if their side lengths are pairwise equal, the distance-preserving property of f maps triangles to congruent triangles. Note that $\{0, 1, e^{i\pi/3}\}$ is an equilateral triangle, so the image $\{f(0), f(1), f(e^{i\pi/3})\} = \{0, 1, f(e^{i\pi/3})\}$ also forms an equilateral triangle.

First we assume $f(e^{i\pi/3}) = e^{i\pi/3}$. For any $z \in \mathbb{C}$, the distance between $f(z)$ and origin must equal to $|z|$, and similarly $|f(z) - 1| = |f - z|$. These two together imply that $f(z)$ is either z or $\overline{z}$. The latter must be false since $e^{i\pi/3}$ does not lie on the real axis and $|e^{i\pi/3} - z| \neq |e^{i\pi/3} - \overline{z}|$ in general. Hence $f(z) = z$, i.e., the identity map. More generally, the original function is simply a rotation by θ.

If $f(e^{i\pi/3}) = e^{-i\pi/3}$ then we compose it with a reflection in the real axis, resulting in the previous case. Thus, a function of this form corresponds to a rotation by θ followed by reflection in the real axis. □

Problem 7: Conway, p.54 problem 1

Find the image of $\{z \in \mathbb{C} : \mathfrak{Re} z < 0, |\mathfrak{Im} z| < \pi\}$ under the exponential function.

Solution. If we write $z = x + iy$ then $e^z = e^x e^{iy}$, so e^x is the modulus and y the argument. That being said, the exponential functions maps vertical lines, on which the real coordinate is fixed, into a circle, since the modulus e^x is fixed, whereas it maps horizontal lines, on which the imaginary coordinate is fixed, into a ray with argument y.

The line $\mathfrak{Re} z = 0$, i.e., the imaginary axis, gets mapped to a circle with radius $e^{\mathfrak{Re} z} = e^0 = 1$. Any line on the left of $\mathfrak{Re} z = 0$ has a negative real component and therefore is inside this circle.

On the other hand, $|\mathfrak{Im} z| < \pi$ represents the collection of all rays with argument in $(-\pi, \pi)$, which corresponds to $\mathbb{C} \backslash (-\infty, 0)$. Taking the intersection we see that the image of our desired set is

$$\{z : |z| < 1\} \backslash ((-1, 0] \times \{0\}).$$