MATH 520 Homework 3

Qilin Ye

February 3, 2022

Problem 1: Alhfors, p.17 problem, modified

Prove that $az + b\overline{z} + c = 0$ determines a line if and only if $|a| = |b| \neq 0$ and $c/b = \overline{c/a}$.

Proof. For $\Rightarrow$, suppose $az + b\overline{z} + c = 0$ determines a line, so in particular there are infinitely many solutions to this equation. Taking conjugate, we also have $\overline{az} + \overline{b}z + \overline{c} = 0$. Multiplying by $\overline{a}, b$ respectively gives

$$|a|^2 z + \overline{a}b\overline{z} + \overline{a}c = |b|^2 z + \overline{a}b\overline{z} + b\overline{c}$$

and subtracting gives

$$z(|a|^2 - |b|^2) = b\overline{c} - \overline{a}c.$$

We see that if one (or both) between $|a| = |b| \neq 0$ and if $c/b = \overline{c}/\overline{a}$ fails, then the above equation has only one solution, so it cannot possibly represent a line. This proves $\Rightarrow$.

Conversely, suppose $|a| - |b| = b\overline{c} - \overline{a}c = 0$ and $az + b\overline{z} + c = 0$. Normalizing the coefficients we have

$$z + (b/a)\overline{z} + c/a = 0$$

with $|b/a| = 1$. Define $k := (b/a)^{1/2}$. Since $(b/a)\overline{k} = k^2\overline{k} = k$, multiplying by $\overline{k}$ we have

$$\overline{k}z + k\overline{z} + c\overline{k}/a = 0.$$

Since

$$\overline{c\overline{k}/a} = k\overline{c/a} = kb\overline{c}/(\overline{a}b) = k\overline{a}c/(\overline{a}b) = kc/b = k^2\overline{k}c/b = \overline{k}bc/(ab) = \overline{k}c/a,$$

we have $C := c\overline{k}/a \in \mathbb{R}$. We can re-express the equation using real-valued constants A, B, such that

$$A \cdot \mathfrak{Re} z + B \cdot \mathfrak{Im} z + C = 0.$$

This corresponds to a line in $\mathbb{R}^2$, and multiplying the second component, i.e., $\mathfrak{Im} z$ by i, we recover the solution set for $az + b\overline{z} + c = 0$, which has to be a line. $\square$

Problem 2: Alhfors, p.96 problem 1

Map the common part of the disks $|z| < 1$ and $|z - 1| < 1$ on the inside of the unit circle. Choose the mapping so that the two symmetries are preserved.

Solution. Note that the intersection is a wedge with vertices $e^{i\pi/3}$ and $e^{-i\pi/3}$. The angle of both vertices is $2\pi/3$. We first construct a Möbius transformation mapping $\infty \to 1, a \to 0$, and $b \to \infty$: this by definition should be $T_1 : z \mapsto (z-a)/(z-b)$. After this, the original wedge becomes an angular sector with the same angle $(2\pi/3)$, bounded by rays corresponding to $-\pi/3$ and $\pi/3$.

Next, we expand the angle from $2\pi/3$ to π, namely, via $T_2 : z \mapsto z^{3/2}$, so that the angular sector becomes the right half plane.

Finally, we use $T_3 : z \mapsto (z-1)/(z+1)$ to map the half plane onto the unit disk. Then $T_3 \circ T_2 \circ T_1$ is the transformation we seek.

Problem 3: Alhfors, p.96 problem 2

Map the region between $|z| = 1$ and $|z - 1/2| = 1/2$ onto the half plane.

Solution. We first consider a mapping $T_1 : z \mapsto 1/(1-z)$. Under this mapping, we have $f(1) = \infty, f(0) = 1, f(-1) = 0.5$, so the real axis maps to the real axis with orientation preserved (since $1/(1-z)$ is Möbius). That is, the upper half plane remains as the upper half plane. We also have $f(1) = \infty, f(i) = 0.5 + 0.5i, f(-1) = 0.5$, so the big circle maps to the line $x = 0.5$, and the big disk maps to the right side of $x = 0.5$. Finally, since $f(1) = \infty, f(0.5 + 0.5i) = 1 + i, f(0) = 0.5$, the small circle maps to $x = 1$ and the small disk to the right side of $x = 1$. Therefore, our original set maps to

$$\{x + yi : x \in [0.5, 1], y \geqslant 0\}.$$

Now we rotate the set by $-\pi/2$, shift everything upward by $3/4$, and stretch by 2π. Namely, define $T_2 : z \mapsto 2\pi(iz + 3/4)$. This way we obtain the image $\{x + yi : x \geqslant 0, y \in [-\pi/2, \pi/2]\}$.

Then we define T_3 using $z \mapsto e^z$ so the set now becomes the right half plane excluding the right half of the unit disk. In order to make this $\mathbb{C}\backslash\mathbb{D}$, we define $T_4 : z \mapsto z^2$. Next, we define $T_5 : z \mapsto z^{-1}$ so the set is now simply $\mathbb{D}$.

It remains to find a mapping from $\mathbb{D}$ to the upper half plane – one such construction is let $f(-i) = 0$, $f(1) = 1$, and $f(i) = \infty$. Using cross ratio we obtain $T_6 = (z, 1, -i, i) = -i(z+i)/(z-i)$. Composing everything and we are done.

Problem 4: Alhfors, p.97 problem 3

Map the complement of the arc $|z| = 1$, $y \geqslant 0$ on the outside of the unit circle so that the points at ∞ correspond to each other.

Solution. We first use $T_1 : z \mapsto z^{-1}$ to map our original set into the top half disk. Using $T_2 : z \mapsto z^2$ we obtain a full disk. Using $T_3 : z \mapsto z^{-1}$ again we obtain our desired result. Note that $T_1(\infty) = 0$, $T_2(0) = 0$, and $T_3(0) = \infty$, so ∞ is preserved.

Problem 6: Conway, p.55 problem 8

If $Tz = \dfrac{az+b}{cz+d}$, show that $T\mathbb{R}_\infty = \mathbb{R}_\infty$ if and only if we can choose a, b, c, d to be real numbers.

Proof. If $a, b, c, d \in \mathbb{R}_\infty$, by closure of addition and multiplication we must have $T\mathbb{R}_\infty = \mathbb{R}_\infty$.
Conversely, suppose $T\mathbb{R}_\infty = \mathbb{R}_\infty$. In particular there exists z_0 with $Tz_0 = 0$, so $az_0 = -b$ and $z_0 = -b/a$. Also, there exists z_∞ with $Tz_1\infty = \infty$, so $cz_\infty = -d$ and $z_\infty = -d/c$. Also, there exists z_1 with $Tz_1 = 1$, so

$$az_1 + b = cz_1 + d \implies z_1(a - c) = d - b = z_0 a - z_\infty c,$$

$$\frac{z_1}{c} - \frac{z_1}{a} = \frac{z_0}{c} - \frac{z_\infty}{a} \implies \frac{z_1 - z_0}{c} = \frac{z_1 - z_\infty}{a} \implies \frac{z_1 - z_0}{z_1 - z_\infty} = \frac{a}{c} \in \mathbb{R}.$$

Since $d/a = (d/c)(c/a)$, we see $d/a \in \mathbb{R}$ as well. It remains to notice that

$$Tz = \frac{az + b}{cd + z} = \frac{z + b/a}{(c/a)z + d/a}$$

and from above, $1, b/a, c/a, d/a$ are all (extended) real numbers. □

Problem 7: Conway, p.55 problem 9

If $Tz = \dfrac{az + b}{cz + d}$, find necessary and sufficient conditions that $T(\Gamma) = \Gamma$ where Γ is the unit circle.

Solution. $T(\Gamma) = \Gamma$ means $|z| = 1 \Rightarrow |T(z)| = 1$ and $|z| = 1 \Rightarrow |\overline{T^{-1}(z)}| = 1$ (since T is assumed to be Möbius — the only non-invertible case clearly fails). Since

$$T(z)\overline{T(z)} = \frac{az + b}{cz + d}\frac{\overline{az} + \overline{b}}{\overline{cz} + \overline{d}}$$

and

$$\begin{aligned}(az + b)(\overline{az} + \overline{b}) - (cz + d)(\overline{cz} + \overline{d}) &= z\overline{z}a\overline{a} + \overline{z}a\overline{b} + zab\overline{b} + b\overline{b} - z\overline{z}c\overline{c} - \overline{z}\overline{c}d - zc\overline{d} - d\overline{d} \\ &= |z|^2(|a|^2 - |c|^2) + z(a\overline{b} - c\overline{d}) + \overline{z}(\overline{a}b + \overline{c}d) + |b|^2 - |d|^2,\end{aligned}$$

one sufficient condition is if $a\overline{b} = c\overline{d}$, $|a|^2 = |c|^2$, and $|b|^2 = |d|^2$. This can be reformulated into $a\overline{b} = c\overline{d}$ and $|a|^2 + |b|^2 = |c|^2 + |d|^2$.
Conversely, if we set $a/\overline{d} = c/\overline{b} = k$, then

$$|a|^2 + |b|^2 = |\lambda|^{-2}|c^2| + |\lambda|^2|d|^2$$

so equality holds only when $|\lambda| = 1$. Hence the sufficient and necessary condition is that $|a| = |b| = |c| = |d|$.