MATH 520 Homework 4

Qilin Ye

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Problem 1: Alhfors, p.108 problem 1

Let γ be the directed line segment from 0 to $1+i$. Comute

$$\int_\gamma x \, \mathrm{d}z.$$

Solution. We use the parametrization $r(t) = t + it$ for $t \in [0,1]$. Then

$$\int_\gamma x \, \mathrm{d}z = \int_0^1 t\gamma'(t) \, \mathrm{d}t = \int_0^1 t(1+i) \, \mathrm{d}t = \frac{1+i}{2}.$$

Problem 2: Alhfors, p.108 problem 2

Compute $\int_{|z|=r} x \, \mathrm{d}z$ for the positive sense of the circle in two ways: first, by parametrization, and second, by observing that

$$x = \frac{z+\overline{z}}{2} = \frac{z + r^2/z}{2}$$

on the circle.

Solution. Using the first approach, we choose the parametrization $r(t) = r\cos t + ir\sin t$ for $t \in [0, 2\pi]$:

$$\begin{aligned}\int_{|z|=r} x \, \mathrm{d}z &= \int_0^{2\pi} r\cos t(-r\sin t + ir\cos t) \, \mathrm{d}t \\ &= \int_0^{2\pi} -r^2\sin(2t)/2 + ir^2(\cos 2t + 1)/2 \, \mathrm{d}t \\ &= \frac{r^2}{2}\int_0^{2\pi} \sin(2t) + i(\cos(2t)+1) \, \mathrm{d}t = \pi r^2 i.\end{aligned}$$

Using the second method,

$$\begin{aligned}\int_{|z|=r} x \, \mathrm{d}z &= \int_{|z|=r} \frac{z + r^2/z}{2} \, \mathrm{d}z \\ &= \int_{|z|=r} \frac{z}{2} \, \mathrm{d}z + \frac{r^2}{2}\int_{|z|=r} \frac{1}{z} \, \mathrm{d}z \\ &= \frac{r^2}{2}\int_{|z|=r} \frac{1}{z} \, \mathrm{d}z = r^2 \cdot 2\pi i/2 = \pi r^2 i.\end{aligned}$$

Problem 3: Alhfors, p.108 problem 3

Compute $\int_{|z|=2} \frac{1}{z^2-1}\,\mathrm{d}z$.

Solution. Using partial fraction decomposition we have

$$\frac{1}{z^2-1} = \frac{1}{(z+1)(z-1)} = \frac{1}{z-1} - \frac{1}{z+1}.$$

Hence

$$\int_{|z|=2} \frac{1}{z^2-1}\,\mathrm{d}z = \int_{|z|=2} \frac{1}{z-1}\,\mathrm{d}z - \int_{|z|=2} \frac{1}{z+1}\,\mathrm{d}z,$$

where both integrals on the RHS are $2\pi i$ by the property of index. Hence the entire integral is 0.

Problem 4: Alhfors, p.108 problem 4

Compute $\int_{|z|=1} |z-1|\,|\mathrm{d}z|$.

Solution. We use the parametrization $r(\theta) = e^{i\theta}, \theta \in [0, 2\pi]$. Then $|r'(\theta)| = |ie^{i\theta}| = 1$ for all θ. Also note that if $z = \cos\theta + i\sin\theta$, then $z-1, z$, and the origin form an isosceles triangle with vertex angle $\theta = \theta/2$. Hence the bottom side, i.e., the vector $z-1$, has length $2\sin(\theta/2)$. Therefore,

$$\int_{|z|=1} |z-1|\,|\mathrm{d}z| = \int_0^{2\pi} 2\sin(\theta/2)\,\mathrm{d}\theta = 8.$$

Problem 5: Alhfors, p.108 problem 5

Suppose that f is analytic on a closed curve γ. Show that

$$\int_\gamma \overline{f(z)} f'(z)\,\mathrm{d}z$$

is purely imaginary.

Proof. Note that

$$\overline{f(z)} f'(z) + \overline{\overline{f(z)} f'(z)} = \overline{f(z)} f'(z) + f(z)\overline{f'(z)}$$

is the exact differential of $f(z)f'(z)$. Therefore

$$\int_\gamma \overline{f(z)} f'(z) + f(z)\overline{f'(z)}\,\mathrm{d}z = \int_\gamma \overline{f(z)} f'(z)\,\mathrm{d}z + \overline{\int_\gamma \overline{f(z)} f'(z)\,\mathrm{d}z} = 0,$$

i.e., our original integral is purely imaginary. □

Problem 6: Alhfors, p.108 problem 6

Suppose that f is analytic and satisfies $|f(z)-1|<1$ on some region Ω. Show that

$$\int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z = 0$$

for every closed curve in Ω.

Proof. Since $\exp(\cdot)$ is injective on Ω, it is well-defined to consider its inverse $\log$. It remains to notice that the integral is the exact differential of $\log(f(z))$ and the claim follows. □

Problem 7: Alhfors, p.108 problem 7

If $P(z)$ is a polynomial and C denotes the circle $|z-a|=R$, show that

$$\int_C P(z)\,\mathrm{d}\overline{z} = -2\pi i R^2 P'(a).$$

Proof. We write

$$P(z) := \sum_{k=0}^{n} a_k (x-z)^k = a_0 + a_1(x-z) + \ldots + a_n(x-z)^n.$$

Also, we choose the parametrization of C by $r(\theta) = Re^{i\theta} + a$ for $\theta \in [0, 2\pi]$. Then, $\mathrm{d}z = iRe^{i\theta}\mathrm{d}\theta$ and $\mathrm{d}\overline{z} = -iRe^{-i\theta}\mathrm{d}\theta$. The rest is just some *algebra plus calculus*:

$$\begin{aligned}
\int_C P(z)\,\mathrm{d}\overline{z} &= \int_0^{2\pi} P(a+Re^{i\theta})(-iRe^{-i\theta})\,\mathrm{d}\theta \\
&= \int_0^{2\pi} \sum_{k=0}^{n} a_k(-Re^{i\theta})^k(-iRe^{-i\theta})\,\mathrm{d}\theta \\
&= \int_0^{2\pi} a_0(-iRe^{-i\theta})\,\mathrm{d}\theta + \sum_{k=1}^{n}\int_0^{2\pi} a_k(-Re^{i\theta})^k(-iRe^{-i\theta})\,\mathrm{d}\theta \\
&= 0 + i\sum_{k=1}^{n}\int_0^{2\pi} a_k(-R)^{k+1}(-e^{ik\theta}e^{-i\theta})\,\mathrm{d}\theta \\
&= i\sum_{k=1}^{n} a_k(-R)^{k+1}\int_0^{2\pi}(-e^{i\theta})e^{-i\theta}\,\mathrm{d}\theta \\
&= -ia_1R^2\int_0^{2\pi} 1\,\mathrm{d}\theta + \sum_{k=2}^{n} 0 = -2\pi i a_1 R^2 = -2\pi i R^2 P'(a).
\end{aligned}$$

□