

MATH 520 Homework 6

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Problem 1: Ahlfors, p.123 p1

Compute

$$\int_{|z|=1} e^z z^{-n} dz, \quad \int_{|z|=2} z^n (1-z)^m dz, \quad \int_{|z|=\rho} |z-a|^{-4} |dz| \quad (\text{where } |a| \neq \rho).$$

Solution. We assume $m, n \in \mathbb{N}$.

For the first integral, if $n = 0$ the integrand is $e^{-n} e^z$, an analytic function. Therefore its integral on a closed curve is 0. If $n \geq 1$, by Cauchy's integral formula,

$$\left. \frac{d^{n-1}}{dz^{n-1}} e^z \right|_{z=0} = \frac{(n-1)!}{2\pi i} \int_{|\zeta|=1} e^\zeta / (\zeta - 0)^{-n} d\zeta$$

so

$$\int_{|\zeta|=1} e^\zeta \zeta^{-n} d\zeta = \frac{2\pi i}{(n-1)!}.$$

For the second one, if $m, n \geq 0$ then the integrand is analytic so the integral is 0.

For the third one, writing z with $|z| = \rho$ as $\rho e^{i\theta}$, we have

$$|dz| = |i\rho e^{i\theta} d\theta| = \rho d\theta = -i \frac{i\rho}{\rho e^{i\theta}} \cdot i\rho e^{i\theta} d\theta = -i\rho \frac{dz}{z}.$$

If $a = 0$ the integral reduces to

$$\int_{|z|=\rho} |z|^{-4} |dz| = -i\rho \cdot \rho^{-4} \int_{|z|=\rho} \frac{1}{z} dz = -i\rho \cdot \rho^{-4} \cdot 2\pi i = \frac{2\pi}{\rho^3}.$$

If $a \neq 0$,

$$\begin{aligned} \int_{|z|=\rho} |z-a|^{-4} |dz| &= -i\rho \int_{|z|=\rho} |z-a|^{-2} |z-a|^{-2} z^{-1} dz \\ &= -i\rho \int_{|z|=\rho} (z-a)^{-2} (\bar{z}-\bar{a})^{-2} z^{-1} dz \\ &= -i\rho \int_{|z|=\rho} (z-a)^2 z^{-2} (\bar{z}-\bar{a})^{-2} z dz \\ &= -i\rho \int_{|z|=\rho} (z-a)^{-2} (z\bar{z} - z\bar{z}/\bar{a})^{-2} z dz \\ &= -\frac{i\rho}{\bar{a}^2} \int_{|z|=\rho} \frac{z}{(z-a)^2 (z-\rho^2/\bar{a})^2} dz. \end{aligned}$$

If $|a| > \rho$, by the formula on derivatives, we have

$$2\pi i \left. \frac{d}{dz} \frac{z}{(z-a)^2} \right|_{z=\rho^2/\bar{a}} =: 2\pi i f'(\rho^2/\bar{a}) = \int_{|z|=\rho} \frac{z}{(z-a)^2 (z-\rho^2/\bar{a})^2} dz.$$

Simplifying gives

$$-\frac{ip}{\bar{a}^2} \cdot 2\pi i \cdot \left. \frac{d}{dz} \frac{z}{(z-a)^2} \right|_{z=\rho^2/\bar{a}} = -\frac{2\pi p}{\bar{a}^2} \frac{\bar{a}^2(a\bar{a} + \rho^2)}{(\rho^2 - a\bar{a})^3} = \frac{2\pi p}{\bar{a}^2} \frac{a + \rho^2/\bar{a}}{(a - \rho^2/\bar{a})^3}. \quad (*)$$

Similarly, if $|a| < \rho$, we have

$$2\pi i \left. \frac{d}{dz} \frac{z}{(z - \rho^2/\bar{a})^2} \right|_{z=a} =: 2\pi i f'(a) = \int_{|z|=\rho} \frac{z}{(z-a)^2(z - \rho^2/\bar{a})^2} dz.$$

Simplifying gives

$$\left. \frac{d}{dz} \frac{z}{(z - \rho^2/\bar{a})^2} \right|_{z=a} = -\frac{a + \rho^2/\bar{a}}{(a - \rho^2/\bar{a})^3}$$

which agrees with the last term in (*). Therefore, for either case, we have

$$\int_{|z|=\rho} |z-a|^{-4} |dz| = \frac{2\pi p}{\bar{a}^2} \frac{a + \rho^2/\bar{a}}{(a - \rho^2/\bar{a})^3}.$$

Problem 2: Alhfors, p.123 p3

If $f(z)$ is analytic and $|f(z)| \leq M$ for $|z| \leq R$, find an upper bound for $|f^{(n)}(z)|$ in $|z| \leq \rho < R$.

Solution. Using Cauchy's integral formula,

$$\begin{aligned} |f^{(n)}(z)| &= \left| \frac{n!}{2\pi i} \int_{\partial D_\rho} \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta \right| = \frac{n!}{2\pi} \left| \int_{|\zeta|=\rho} \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta \right| \\ &\leq \frac{n!}{2\pi} \int_{|\zeta|=\rho} \frac{|f(\zeta)|}{|(\zeta - z)^{n+1}|} d\zeta \leq \frac{n!}{2\pi} \int_{|\zeta|=\rho} \frac{M}{|\zeta - z|^{n+1}} d\zeta \\ &\leq \frac{n!}{2\pi} \int_{|\zeta|=\rho} \frac{M}{(\rho - |z|)^{n+1}} d\zeta = \frac{Mn!\rho}{(\rho - |z|)^{n+1}}. \end{aligned}$$

Problem 3: AlhFors, p.123 p5

Show that successive derivatives of an analytic function can never satisfy $f^{(n)}(z) > n!n^n$. Formulate a sharper theorem of the same kind.

Proof. Let r be sufficiently small so that f is analytic on $D_r(z)$. Then on the disk f is bounded by, say M , and

$$|f^{(n)}(z)| = \frac{n!}{2\pi} \left| \int_{|\zeta-z|=r} \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta \right| \leq \frac{n!}{2\pi} \int_{|\zeta-z|=r} \frac{M}{|\zeta - z|^{n+1}} d\zeta = \frac{Mn!r}{r^{n+1}} = \frac{Mn!}{r^n}.$$

Since

$$\limsup_{n \rightarrow \infty} \frac{Mn!/r^n}{n^n n!} = \limsup_{n \rightarrow \infty} \frac{M}{n^n r^n} = 0 \implies \frac{Mn!}{r^n} \in \mathcal{O}(n^n n!) \setminus \Theta(n^n n!).$$

we see that $f^{(n)}(z)$ must be dominated by $n!n^n$ for large n . For a sharper bound we can consider $(n!)^2$ where the limsup defined above still equals 0. □

Problem 4: Alhfors p.129 p1

If $f(z), g(z)$ have algebraic orders h and k at $z = a$, show that fg has order $h + k$, f/g has $h - k$, and $f + g$ has an order which does not exceed $\max\{h, k\}$.

Proof. By definition of algebraic orders on page 128, if f, g have orders h, k at a , then

$$\lim_{z \rightarrow a} |z - a|^\alpha |f(z)| = 0 \quad \lim_{z \rightarrow a} |z - a|^\beta |g(z)| = 0$$

if and only if $\alpha \geq h, \beta \geq k$, respectively. Taking product gives

$$\lim_{z \rightarrow a} |z - a|^p |f(z)g(z)| = 0 \iff p \geq h + k,$$

which implies $h + k$ is the order of fg .

For quotient,

$$\lim_{z \rightarrow a} |z - a|^p |f(z)/g(z)| = \lim_{z \rightarrow a} |z - a|^{p-(h-k)} |z - a|^{h-k} |f(z)/g(z)|$$

for all $p \geq h - k$. If it has order $< h - k$, then $(f/g)g$ has an order less than $h - k + k = h$, contradicting our assumption.

Finally,

$$\lim_{z \rightarrow a} |z - a|^p |f(z) + g(z)| \leq \lim_{z \rightarrow a} |z - a|^p |f(z)| + \lim_{z \rightarrow a} |z - a|^p |g(z)|$$

so if $p \geq \max\{h, k\}$, both terms on the RHS are guaranteed to have limit 0. This means that the order is at most the max. \square

Problem 5: Alhfors, p.130 p2

Show that a function which is analytic in the whole plane and has a nonessential singularity at ∞ reduces to a polynomial.

Proof. Since f cannot have a removable singularity at ∞ , by assumption it must be a pole. Therefore $f(1/z)$ has a pole at the origin. By a theorem shown in class, there exists g analytic with $g(0) \neq 0$ and some $n \in \mathbb{N}$ such that

$$f(1/z) = \frac{g(1/z)}{z^n}.$$

Since $g(1/z)$ is analytic, it is locally bounded around the origin, so there exists r and M such that $\sup |g(1/z)| \leq M$ on $D_r(0)$, so

$$|f(1/z)| \leq \frac{M}{|z|^n} \quad \text{for } |z| \leq r.$$

Equivalently,

$$|f(z)| \leq \frac{M}{|z|^n} \quad \text{for } |z| \geq 1/r.$$

By a proposition shown in class (right after Cauchy estimate), this implies that $f^{(n+1)}(z) = 0$, so f is a polynomial of degree $\leq n$. \square

Problem 6: Alhfors, p.130 p4

Show that any function which is meromorphic in the extended plane is rational.

Proof. If f has a pole at ∞ , it can only have finitely many poles for poles are isolated. Call these z_1, \dots, z_n with orders e_1, \dots, e_n . Then

$$g(z) := \prod_{i=1}^n (z - z_i)^{e_i} f(z)$$

only has a pole at ∞ . By the previous part g must be a polynomial. Dividing by $\prod_{i=1}^n (z - z_i)^{e_i}$ and we are done.

If f does not have a pole at ∞ , by assumption it cannot be an essential singularity, so it is again bounded and we can apply Cauchy estimate to obtain the result. \square

Problem 7: Alhfors, p.130 p6

Show that an isolated singularity of $f(z)$ cannot be a pole of $\exp f(z)$.

Proof. If $f(z)$ has a removable singularity then by definition f is locally bounded. Therefore $\exp f$ is also locally bounded, showing that $\exp f$ is removable and therefore not a pole.

If $f(z)$ is an essential singularity then by theorem 9, for $c \in \mathbb{C}$, for any neighborhood of z , there exists points that gets arbitrarily close to c . Therefore z cannot be a pole.

Finally assume z is a pole of f . Suppose its order is $n \in \mathbb{N}$ so that f' has a pole at z with order $n + 1$. Suppose $\exp f$ also has a pole at z with order m and its derivative, $f' \exp f$ has a pole at z with order $m + 1$. But then this implies $m + 1 = n + 1 + m$ so $n = 0$, contradiction. Therefore $\exp f$ cannot share any pole with f . \square