MATH 520 Homework 6

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Problem 1: Alhfors, p.123 p1

Compute

$$\int_{|z|=1} e^z z^{-n}\,\mathrm{d}z, \qquad \int_{|z|=2} z^n(1-z)^m\,\mathrm{d}z, \qquad \int_{|z|=\rho}|z-a|^{-4}|\mathrm{d}z| \quad (\text{where } |a|\neq\rho)$$

where γ is the directed line segment from 0 to $1+i$.

Solution. We assume $m,n\in\mathbb{N}$.

For the first integral, if $n=0$ the integrand is $e^{-n}e^z$, an analytic function. Therefore its integral on a closed curve is 0. If $n\geqslant 1$, by Cauchy's integral formula,

$$\frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}e^z\bigg|_{z=0} = \frac{(n-1)!}{2\pi i}\int_{|\zeta|=1} e^\zeta/(\zeta-0)^{-n}\,\mathrm{d}\zeta$$

so

$$\int_{|\zeta|=1} e^\zeta\zeta^{-n}\,\mathrm{d}\zeta = \frac{2\pi i}{(n-1)!}.$$

For the second one, if $m,n\geqslant 0$ then the integrand is analytic so the integral is 0.

For the third one,

Problem 2: Alhfors, p.123 p3

If $f(z)$ is analytic and $|f(z)|\leqslant M$ for $|z|\leqslant R$, find an upper bound for $|f^{(n)}(z)|$ in $|z|\leqslant\rho<R$.

Solution. Using Cauchy's integral formula,

$$\begin{aligned}|f^{(n)}(z)| &= \left|\frac{n!}{2\pi i}\int_{\partial D_\rho}\frac{f(\zeta)}{(\zeta-z)^{n+1}}\,\mathrm{d}\zeta\right| = \frac{n!}{2\pi}\left|\int_{|\zeta|=\rho}\frac{f(\zeta)}{(\zeta-z)^{n+1}}\,\mathrm{d}\zeta\right| \\ &\leqslant \frac{n!}{2\pi}\int_{|\zeta|=\rho}\frac{|f(\zeta)|}{|(\zeta-z)^{n+1}|}\,\mathrm{d}\zeta \leqslant \frac{n!}{2\pi}\int_{|\zeta|=\rho}\frac{M}{|\zeta-z|^{n+1}}\,\mathrm{d}\zeta \\ &\leqslant \frac{n!}{2\pi}\int_{|\zeta|=\rho}\frac{M}{(\rho-|z|)^{n+1}}\,\mathrm{d}\zeta = \frac{Mn!\rho}{(\rho-|z|)^{n+1}}.\end{aligned}$$

Problem 3: AlhFors, p.123 p5

Show that successive derivatives of an analytic function can never satisfy $f^{(n)}(z)>n!n^n$. Formulate a shaper theorem of the same kind.

Proof. Let r be sufficiently small so that f is analytic on $D_r(z)$. Then on the disk f is bounded by, say M, and

$$|f^{(n)}(z)| = \frac{n!}{2\pi}\left|\int_{|\zeta-z|=r}\frac{f(\zeta)}{(\zeta-z)^{n+1}}\,\mathrm{d}\zeta\right| \leqslant \frac{n!}{2\pi}\int_{|\zeta-z|=r}\frac{M}{|\zeta-z|^{n+1}}\,\mathrm{d}\zeta = \frac{Mn!r}{r^{n+1}} = \frac{Mn!}{r^n}.$$

Since

$$\limsup_{n\to\infty}\frac{Mn!/r^n}{n^n n!} = \limsup_{n\to\infty}\frac{M}{n^n r^n} = 0 \implies \frac{Mn!}{r^n} \in \mathcal{O}(n^n n!)\backslash\Theta(n^n n!).$$

we see that $f^{(n)}(z)$ must be dominated by $n!n^n$ for large n. For a shaper bound we can consider $(n!)^2$ where the limsup defined above still equals 0. □

Problem 4: Alhfors p.129 p1

If $f(z), g(z)$ have algebraic orders h and k at $z = a$, show that fg has order $h+k$, f/g has $h-k$, and $f+g$ has an order which does not exceed $\max\{h,k\}$.

Proof. By definition of algebraic orders on page 128, if f, z have orders h, k at a, then

$$\lim_{z\to a}|z-a|^{\alpha}|f(z)| = 0 \qquad \lim_{z\to a}|z-a|^{\beta}|g(z)| = 0$$

if and only if $\alpha \geqslant h$, $\beta \geqslant k$, respectively. Taking product gives

$$\lim_{z\to a}|z-a|^{p}|f(z)g(z)| = 0 \iff p \geqslant h+k,$$

which implies $h+k$ is the order of fg.

For quotient,

$$\lim_{z\to a}|z-a|^{p}|f(z)/g(z)| = \lim_{z\to a}|z-a|^{p-(h-k)}|z-a|^{h-k}|f(z)/g(z)|$$

for all $p \geqslant h-k$. If it has order $< h-k$, then $(f/g)g$ has an order less than $h-k+k = h$, contradicting our assumption.

Finally,

$$\lim_{z\to a}|z-a|^{p}|f(z)+g(z)| \leqslant \lim_{z\to a}|z-a|^{p}|f(z)| + \lim_{z\to a}|z-a|^{p}|g(z)|$$

so if $p \geqslant \max\{h,k\}$, both terms on the RHS are guaranteed to have limit 0. This means that the order is at most the max. □

Problem 5: Alhfors, p.130 p2

Show that a function which is analytic in the whole plane and has a nonessential singularity at ∞ reduces to a polynomial.

Proof. Since f cannot have a removable singularity at ∞, by assumption it must be a pole. Therefore $f(1/z)$ has a pole at the origin. By a theorem shown in class, there exists g analytic with $g(0) \neq 0$ and some $n \in \mathbb{N}$ such that

$$f(1/z) = \frac{g(1/z)}{z^n}.$$

Since $g(1/z)$ is analytic, it is locally bounded around the origin, so there exists r and M such that $\sup g(1/z) \leqslant M$

on $D_r(0)$, so

$$|f(1/z)| \leqslant \frac{M}{|z|^n} \qquad \text{for } |z| \leqslant r.$$

Equivalently,

$$|f(z)| \leqslant \frac{M}{|z|^n} \qquad \text{for } |z| \geqslant 1/r.$$

By a proposition shown in class (right after Cauchy estimate), this implies that $f^{(n+1)}(z) = 0$, so f is a polynomial of degree $\leqslant n$. □

Problem 6: Alhfors, p.130 p4

Show that any function which is meromorphic in the extended plane is rational.

Proof. If f has a pole at ∞, it can only have finitely many poles for poles are isolated. Call these $z_1, \ldots, z_n$ with orders $e_1, \ldots, e_n$. Then

$$g(z) := \prod_{i=1}^{n} (z - z_i)^{e_i} f(z)$$

only has a pole at ∞. By the previous part g must be a polynomial. Dividing by $\prod_{i=1}^{n}(z - z_i)^{e_i}$ and we are done.
If f does not have a pole at ∞, by assumption it cannot be an essential singularity, so it is again bounded and we can apply Cauchy estimate to obtain the result. □

Problem 7: Alhfors, p.130 p6

Show that an isolated singularity of $f(z)$ cannot be a pole of $\exp f(z)$.

Proof. If $f(z)$ has a removable singularity then by definition f is locally bounded. Therefore $\exp f$ is also locally bounded, showing that $\exp f$ is removable and therefore not a pole.
If $f(z)$ is an essential singularity then by theorem 9, for $c \in \mathbb{C}$, for any neighborhood of z, there exists points that gets arbitrarily close to c. Therefore z cannot be a pole.
Finally assume z is a pole of f. Suppose its order is $n \in \mathbb{N}$ so that f' has a pole at z with order $n + 1$. Suppose $\exp f$ also has a pole at z with order m and its derivative, $f' \exp f$ has a pole at z with order $m + 1$. But then this implies $m + 1 = n + 1 + m$ so $n = 0$, contradiction. Therefore $\exp f$ cannot share any pole with f. □