MATH 520 Homework 7

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Problem 1: Alhfors, page 130 p3

Show that $e^z, \sin z$, and $\cos z$ have essential singularities at ∞.

Proof. By a result from the previous homework, if these functions have oo as a nonessential singularity then they must be polynomials. However, they all have infinitely many zeros so they cannot be polynomials. □

Problem 2: Alhfors, page 130 p5

Prove that an isolated singularity of $f(z)$ is removable as soon as either $\mathfrak{Re} f$ or $\mathfrak{Im} f$ is bounded from below.

Proof. Suppose WLOG that $\mathfrak{Re} f$ is bounded from below by c. Then the image of f is a subset of $\{z \in \mathbb{C} : \mathfrak{Re} z > c\}$. Let T be the composition of $z \mapsto z - c$ with $z \mapsto (z-1)/(z+1)$ so that the image of $T \circ f$ is (a subset of) the unit disk.

Let z_0 be an isolated singularity of f. Then $T(f(z_0)) \in \mathbb{D}$ and is locally bounded. Thus, z_0 is a removable singularity for $T \circ f$. Let $D_r(z_0)$ be some neighborhood of z_0 in which z_0 is the only singularity of f (or of $T \circ f$). Then $|T \circ f(z)| < 1$ on $D_r(z_0)$. Let g be the extension of $T \circ f$. By the maximum principle, $g(z_0) < 1$ as well. Therefore there exists a sufficiently small neighborhood $D_{r_0}(g(z_0))$ around $g(z_0)$ still contained in $\mathbb{D}$. The inverse image under T^{-1} gives us a neighborhood around z_0, on which f is bounded. This proves the claim. □

Problem 3: Alhfors, page 133 p1

Determine explicitly the largest disk about the origin whose image under the mapping $w = z^2 + z$ is one to one.

Solution. Define $f(z) := z^2 + z$. Then $f'(z) = 2z + 1$ so $f'(-1/2) = 0$. This means that the radius must $< 1/2$. On the other hand, if $|z_1|, |z_2| < 1/2$ then $f(z_1) \neq f(z_2)$, since

$$z_1^2 + z_1 = z_2^2 + z_2 \implies (z_1 - z_2)(z_1 + z_2) = z_1^2 - z_2^2 = -(z_1 - z_2) \implies z_1 + z_2 = -1$$

which violates the triangle inequality. Hence the answer is simply $D_{1/2}(0)$.

Problem 4: Alhfors, page 133 p2

Do the same for $w = e^z$?

Solution. Since $e^{2\pi i} = 1$, we cannot have two points in the disk whose imaginary part has difference 2π. Thus the radius $< \pi$. On the other hand, in $D_\pi(0)$, if $e^{x_1+iy_1} = e^{x_2+iy_2}$, then $e^{x_1} = e^{x_2}$ implies $x_1 = x_2$, and $e^{iy_1} = e^{iy_2}$ happens only when $y_1 = y_2$, given that the points are in $D_\pi(0)$. Hence $D_\pi(0)$ is the largest disk on which $\exp$ is injective.

Problem 5: Alhfors, page 133 p3

Apply the representation $f(z) = w_0 + \zeta(z)^n$ to $\cos z$ with $z_0 = 0$. Determine $\zeta(z)$ explicitly.

Solution. Since $\cos 0 = 1$ we have $w_0 = 1$. Also, since $\cos z - 1$ has order 2 at the origin, $n = 2$. Then $f(z) - w_0 = \cos z - 1 = \cos(2 \cdot z/2) - 1 = -2\sin^2(z/2)$. Thus $\zeta(z) := i\sqrt{2}\sin(z/2)$ is the one we seek. If we were to factor out the z's, define

$$h(z) := \begin{cases} \zeta(z)/z & z \neq 0 \\ i/\sqrt{2} & z = 0 \end{cases}$$

(so that h is analytic) and we obtain

$$f(z) - f(z_0) = (z - z_0)^2 h(z)^2 \qquad h(z_0) \neq 0.$$