MATH 520 Homework 8

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Problem 1: Alhfors, 136/2

If $f(z)$ is analytic and $\mathfrak{Im} f(z) > 0$ for $\mathfrak{Im} z > 0$, show that

$$\frac{|f(z) - f(z_0)|}{|f(z) - \overline{f(z_0)}|} \leqslant \frac{|z - z_0|}{|z - \overline{z_0}|} \qquad \text{and} \qquad \frac{|f'(z)|}{\mathfrak{Im} f(z)} \leqslant \frac{1}{y} \text{ where } z = x + iy.$$

Proof. The claim clearly fails if we don't assume $\mathfrak{Im} z \geqslant 0$. For example consider $f(z) := \exp(z)$ with small x and a sequence of y increasing with period 2π. Thus we assume z, z_0 are both in the upper half plane.

Define $T : z \mapsto (z - z_0)/(z + z_0)$ and $S : z \mapsto (z - f(z_0))/(z - \overline{f(z_0)})$. Both map the upper half plane to $\mathbb{D}$. The first maps z_0 to 0 and the second maps $f(z_0)$ to 0. Then

$$\varphi : S \circ f \circ T^{-1}$$

is a mapping from $\mathbb{D}$ to $\mathbb{D}$ with $\varphi(0) = 0$, so Schwarz implies

$$|S(f(T^{-1}(z)))| \leqslant |z| \qquad \text{for all } z \in \mathbb{D}$$

so

$$|S(f(z))| \leqslant |T(z)| \qquad \text{for all } \mathfrak{Im} z \geqslant 0.$$

This recovers the claim.

For the second one, if $y = 0$ the claim is trivial. If $y > 0$, this is equivalent to

$$|f'(z)| \leqslant \frac{\mathfrak{Im} f(z)}{\mathfrak{Im} z}.$$

Form the previous part, for sufficiently small h so that $\mathfrak{Im}(z + h) > 0$, we have

$$\frac{|f(z+h) - f(z)|}{|f(z+h) - \overline{f(z)}|} \leqslant \frac{|(z+h) - z|}{|z + h - \overline{z}|} \implies \frac{|f(z+h) - f(z)|}{h} \leqslant \frac{|f(z+h) - \overline{f(z)}|}{|z + h - \overline{z}|}.$$

Let $h \to 0$. The limit of the LHS is $|f'(z)|$, whereas the limit of the RHS is, by continuity,

$$\frac{|f(z) - \overline{f(z)}|}{z - \overline{z}} = \frac{|2\mathfrak{Im} f(z)|}{|2\mathfrak{Im} z|} = \frac{\mathfrak{Im} f(z)}{\mathfrak{Im} z},$$

and we are done. □

Problem 2: Alhfors, 136/5

Prove by Schwarz's lemma that every one-to-one conformal mapping of a disk onto another (or half plane) is given by a linear transformation.

Proof. If $f : \mathbb{D} \to \mathbb{D}$ is conformal, let $c := f(0)$ so that the mapping $T : z \mapsto (z-c)/(1-z\bar{c})$ is from $\mathbb{D}$ to $\mathbb{D}$, so $g := T \circ f$ is from $\mathbb{D}$ to $\mathbb{D}$ with 0 mapped to 0. By Schwarz lemma $|g(z)| \leqslant |z|$. Conversely $g^{-1} = f^{-1} \circ T^{-1}$ satisfies the same property so $|g(z)| \leqslant |z| \leqslant |g^{-1}(z)|$. Nesting gives

$$|z| = |g^{-1}(g(z))| \leqslant |g(z)| \leqslant |z|$$

so the second condition of Schwartz lemma is attained and we see $g(z)$ must be of form $g(z) = cz$ with $|c| = 1$. Therefore $f = g \circ f^{-1}$ is Möbius.

For general regions, we need to only replace $T \circ f$ by some corresponding stretching factors and need not to worry about other things. □

Problem 3: Conway, 132/1

Suppose $|f(z)| \leqslant 1$ for $|z| < 1$ and f is non-constant analytic. By considering $g : \mathbb{D} \to \mathbb{D}$ given by

$$g(z) := \frac{f(z) - a}{1 - \bar{a} f(z)}$$

where $a = f(0)$, show that

$$\frac{|f(0)| - |z|}{1 + |f(0)||z|} \leqslant |f(z)| \leqslant \frac{|f(0)| + |z|}{1 - |f(0)||z|}$$

for $|z| < 1$.

Proof. The g as defined above also satisfies $g(0) = 0$. Using Schwarz lemma we have

$$|g(z)| \leqslant |z| \implies |f(z) - a| \leqslant |1 - \bar{a} f(z)||z|.$$

Using triangle inequality we have

$$|f(z)| - |a| \leqslant |z| + |a||f(z)||z|$$

which, upon rearranging, gives

$$|f(z)| - |f(z)||a||z| \leqslant |a| + |z| \implies |f(z)| \leqslant \frac{|f(0)| + |z|}{1 - |f(0)||z|}.$$

The other side of the inequality is by replacing $|f(z)| - |a| \leqslant |f(z) - a|$ by $|a| - |f(z)| \leqslant |f(z) - a|$. □

Problem 4: Conway, 132/2

Does there exist an analytic function $f : \mathbb{D} \to \mathbb{D}$ with $f(1/2) = 3/4$ and $f'(1/2) = 2/3$?

Solution. No. This is a direct violation of the Schwarz-Pick lemma:

$$\frac{2/3}{1-(3/4)^2} = \frac{2/3}{7/16} = \frac{32}{21} > \frac{4}{3} = \frac{1}{1-(1/2)^2}.$$

Problem 5: Conway, 133/6

Let f be analytic in $\mathbb{D}$ and suppose that $|f(z)| \leqslant M$ for all $z \in \mathbb{D}$.

(1) If $f(z_k) = 0$ for $1 \leqslant k \leqslant n$ show that

$$|f(z)| \leqslant M \prod_{k=1}^{n} \frac{|z-z_k|}{|1-\overline{z_k}z|}$$

for $|z| < 1$.i

(2) If $f(z_k) = 0$ for $1 \leqslant k \leqslant n$ and $z_k \neq 0$ and $f(0) = Me^{ia} z_1 \ldots z_n$, find a formula of f.

Proof. (1) Consider

$$Tz := \prod_{k=1}^{n} \frac{z-z_k}{1-\overline{z_k}z}$$

which maps $\mathbb{D}$ to $\mathbb{D}$ and $\partial\mathbb{D}$ to $\partial\mathbb{D}$.Then f/T defined on $\mathbb{D}$ is analytic and by Maximum Principle attains maximum modulus only on $\partial\mathbb{D}$. Hence

$$\sup_{z\in\mathbb{D}} |f(z)/T(z)| = \sup_{|z|=1} |f(z)/T(z)| \leqslant \sup_{|z|=1} |f(z)| = M.$$

Multiplying both sides by $T(z)$ and we obtain our result.

(2)

$$f(z) = Me^{ia} \prod_{k=1}^{n} \frac{z_k - z}{1-\overline{z_k}z}.$$

□

Problem 6: Conway, 133/6

Suppose f is analytic in some region containing $\overline{\mathbb{D}}$ and $|f(z)| = 1$ for $|z| = 1$. Find a formula for f.

Solution. If f has no zeros in $\mathbb{D}$, then $1/f$ is analytic. By Maximum principle on both f and $1/f$ we are forced to have $f \equiv z$ for some $z \in \partial\mathbb{D}$.

If on the other hand f has zeros in $\mathbb{D}$ then, since no zero can be a limit point, there can only be finitely many zeros, say $z_1, \ldots, z_n$ with multiplicities $c_1, \ldots, c_n$. Then there exists g analytic in $\mathbb{D}$ with

$$f(z) = g_n \cdot \prod_{i=1}^{n} \left(\frac{z-z_i}{1-\overline{z_i}z} \right)^{c_i}$$

where g has no zero, maps $\mathbb{D}$ to $\mathbb{D}$ and $\partial\mathbb{D}$ to $\partial\mathbb{D}$. Using the previous part we see that $g_n \equiv \lambda$ for some $\lambda \in \partial\mathbb{D}$. That is,

$$f(z) = \lambda \prod_{i=1}^{n} \left(\frac{z-z_i}{1-\overline{z_i}z} \right)^{c_i}.$$

Problem 7: Conway, 133/8

Is there an analytic function f on $\mathbb{D}$ such that $|f(z)| < 1$ for $|z| < 1$, $f(0) = 1/2$, and $f'(0) = 3/4$?

Solution. If f exists, and if we define $T : z \mapsto (z - 1/2)/(1 - z/2)$, then $T(1/2) = 0$, so $T(f(0)) = 0$ and of course $|T(f(z))| < 1$ for $|z| < 1$. Furthermore,

$$(T \circ f)'(0) = T'(1/2)f'(0) = \frac{3}{(2 - 0.5)^2} \cdot \frac{3}{4} = 1$$

so Schwarz implies that there exists c such that $T(f(z)) = cz$, so $f(z) = T^{-1}(cz)$, i.e.,

$$f(z) = \frac{cz + 1/2}{1 + cz/2} \implies f'(z) = \frac{3c}{(cz + 2)^2},$$

so $f'(0) = 3/4$ implies $c = 1$. Conversely, we check that $f(z)$ defined above with $c = 1$ is indeed a solution. Thus we are done proving existence and uniqueness.