MATH 520 Homework 9

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Problem 1: Alhfors, page 148 p1

Prove that $p\,\mathrm{d}x + q\,\mathrm{d}y$ is locally exact in Ω if and only if

$$\int_{\partial R} p\,\mathrm{d}x + q\,\mathrm{d}y = 0$$

for every rectangle $R \subset \Omega$ with sides parallel to the axes.

Proof. First assume the integral condition is satisfied. For $z \in \Omega$, let $B \subset \Omega$ be a ball containing z and pick $z_0 \in B$. Define

$$U(z; z_0) := \int_\gamma p\,\mathrm{d}x + q\,\mathrm{d}y$$

where γ is a path starting from z_0, moving horizontally then vertically, eventually reaching z (so that U is a function of z only and z_0 is fixed). It follows from what we have shown in class that the x, y partials of U are p and q, respectively, so the 1-form is locally exact.

Conversely, suppose $p\,\mathrm{d}x + q\,\mathrm{d}y$ is locally exact and let R be given. Each $z \in R$ is locally exact so there exists some ball $B_z \subset \Omega$ containing z in which the 1-form is exact. Consider the cover $\{B_z\}_{z\in R}$, an open cover for R, which by compactness has a finite subcover $\{B_{z_i}\}_{i=1}^n$. By the Lebesgue covering lemma there exists $\delta > 0$ such that for all $z \in R$, $B(z, \delta)$ is contained in some B_{z_i}. Now divide R into sufficiently small meshes, e.g, with small rectangles with side lengths $< \delta/100$. Then

$$\int_{\partial R} p\,\mathrm{d}x + q\,\mathrm{d}y = \sum_{R_i \in \text{ mesh}} \int_{\partial R_i} p\,\mathrm{d}x + q\,\mathrm{d}y = 0.$$ □

Problem 2: Alhfors, page 148 p2

Prove that a region obtained from a simply connected region by removing m points has connectivity $m + 1$ and find a homology basis.

Proof. Let the original simply connected region be Ω and let the points be $a_1, \ldots, a_m$, respectively. Call the new region E. Then $\mathbb{C}_\infty \backslash E = (\mathbb{C}_\infty \backslash E) \cup \{a_1, \ldots, a_m\}$, which is indeed the disjoint union of $m + 1$ sets.

For each a_i, define γ_i to be the boundary of a disk centered at a_i with sufficiently small radius so that γ_i and γ_j do not intersect for any $i \neq j$. □

Problem 3: Alhfors, page 148 p4

Show that the single-valued analytic branches of $\log z$, z^α, and z^z can be defined in any simply connected region which does not contain the origin.

Proof. This follows from a corollary mentioned in class. If $f(z) = z$ then in the region described by the problem, $f(z) \neq 0$. Hence $\log f(z) = \log z$ is well-defined. Then z^α and z^z are simply $\exp(\alpha \log z)$ and $\exp(z \log z)$, respectively. □

Problem 4: Alhfors, page 158 p5

Show that a single-valued analytic branch of $\sqrt{1-z^2}$ can be defined in any region such that the points ± 1 are in the same component of the complement. What are the possible values of

$$\int \frac{1}{\sqrt{1-z^2}}\,\mathrm{d}z$$

over a closed curve in the region?

Proof. By assumption, if γ is a closed curve in Ω then $n(\gamma, 1) = n(\gamma, -1)$. Therefore

$$\int_\gamma \frac{1}{z-1} - \frac{1}{z+1}\,\mathrm{d}z = n(\gamma,1) - n(\gamma,-1) = 0$$

for all closed curve γ, i.e., $1/(z-1) - 1/(z+1)$ has a primitive on Ω. Therefore the above has a primitive f. Define $g(z) := (z+1)\exp(f(z)/2)$. If we define $h(z) := \exp(f(z))(z+1)/(z-1)$, then

$$h'(z) = \exp(f(z))f'(z)\frac{z+1}{z-1} + \exp(f(z)) \cdot \frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{z+1}{z-1}\right] = h(z)\left[f'(z) + \frac{1}{z+1} - \frac{1}{z-1}\right] = 0.$$

Therefore h is constant so

$$\frac{g(z)^2}{h(z)} = \frac{(z+1)^2\exp(f(z))}{\exp(f(z))(z+1)/(z-1)} = (z^2-1),$$

and $g(z)/\sqrt{h(z)}$ is a holomorphic branch.

For the second part, note that $1/\sqrt{1-z^2}$ is analytic at 0. Therefore for a closed curve including the origin and for $r < 1$, we have

$$\int_{|z|=r} \frac{1}{\sqrt{1-z^2}}\,\mathrm{d}z = \int_{|z|=r} \frac{1}{z\sqrt{(1/z)^2-1}}\,\mathrm{d}z = -\int_{|w|=1/r} \frac{1}{w\sqrt{w^2-1}}\,\mathrm{d}w$$

which by Cauchy equals $2\pi i n(\gamma, 0)/i = 2\pi n(\gamma, 0)$. Therefore the possible values are integer multiples of 2π. □

Problem 5: Conway, page 87 p1

Suppose $f : G \to G$ is analytic and define $\varphi : G \times G \to \mathbb{C}$ by $\varphi(z,w) = ((f(z) - f(w))/(z-w)$ if $z \neq w$ and $\varphi(z,z) = f'(z)$. Prove φ is continuous and for each fixed w, $z \mapsto \varphi(z,w)$ is analytic.

Proof. Continuity on $G \times G$ except when $z = w$ is trivial and follows from definition when both arguments equal. Also, with w fixed, the analyticity of $z \mapsto \varphi(z,w)$ is obvious.

For the case $z = w$, note that $f(z)$ has a Taylor series $f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(w)}{k!}(z-w)^k$, so

$$\frac{f(z) - f(w)}{z - w} = 0 + \sum_{k=1}^{\infty} \frac{f^{(k)}(w)}{k!}(z-w)^{k-1} = f'(w) + \mathcal{O}((z-w)).$$

As $z \to w$, we recover the limit $f'(w)$. □

Problem 6: Conway, page 87 p8

Let G be a region and suppose $f_n : G \to \mathbb{C}$ is analytic for each $n \geqslant 1$. Suppose that $\{f_n\}$ converges uniformly to a function $f : G \to \mathbb{C}$. Show that f is analytic.

Proof. For each z inside the interior of G, let $D \subset G$ be a disk containing that point with a center $z_0 \neq z$. Let its boundary be γ. Then

$$f_n(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} \,\mathrm{d}\zeta \qquad \text{and } f(z) = \lim_{n\to\infty} f_n(z) = \int_\gamma \frac{f(\zeta)}{\zeta - z} \,\mathrm{d}\zeta$$

since γ is compact and the limit and integral sign interchange. Hence, for any z_0,

$$\begin{aligned}\frac{f(z) - f(z_0)}{z - z_0} &= \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{z - z_0} \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - z_0} \right) \mathrm{d}\zeta \\ &= \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)(\zeta - z_0)} \,\mathrm{d}\zeta \to \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^2} \,\mathrm{d}\zeta\end{aligned}$$

as $z \to z_0$. Therefore f is analytic with

$$f'(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^2} \,\mathrm{d}\zeta,$$

as expected. □

Problem 7: Conway, page 87 p9

Show that if $f : \mathbb{C} \to \mathbb{C}$ is a continuous function such that f is analytic off $[-1, 1]$ then f is entire.

Proof. By Cauchy theorem, it suffices to show that the integral of f over the boundary of any upright triangle is 0. Let R be any such rectangle. If $R \cap [-1, 1] = \varnothing$ then this follows from analyticity, so it remains to consider the other case.

Suppose R does intersect $[-1, 1]$ and let $\epsilon > 0$ be small so that we can form two rectangles by removing the part $\{z \in R : |\mathfrak{Im} z| < \epsilon\}$. Let the upper part be $R_{\epsilon,1}$ and the lower be $R_{\epsilon,2}$. It follows that f is analytic on them both so

$$\int_{\partial R_{\epsilon,1}} f \,\mathrm{d}z = \int_{\partial R_{\epsilon,2}} f \,\mathrm{d}z = 0.$$

Now let $\epsilon \to 0$. Since f is continuous and R is compact, f is uniformly continuous on R, so indeed the integrals converge. That is,

$$\lim_{\epsilon\to 0} \int_{\partial R_{\epsilon,1}} f \,\mathrm{d}z = \int_{R_1} f \,\mathrm{d}z \qquad \text{where } R_1 := \{z \in R : \mathfrak{Im} z \geqslant 0\}$$

and likewise

$$\lim_{\epsilon\to 0} \int_{\partial R_{\epsilon,2}} f \,\mathrm{d}z = \int_{R_2} f \,\mathrm{d}z \qquad \text{where } R_2 := \{z \in R : \mathfrak{Im} z \leqslant 0\}.$$

Adding R_1 and R_2 we obtain R, so the integral of f over R is 0, proving the claim. □