MATH 520 Midterm I

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Problem 1

(a) Map the domain

$$\Omega := \{z \in \mathbb{C} : |z| < 1, |z + 1/2| > 1/2\}$$

conformally onto $\mathbb{D}$.

(b) Map the domain

$$\Omega := \{z \in \mathbb{C} : |z| < 1, \mathfrak{Im} z > 0\}$$

conformally onto $\mathbb{D}$.

Solution. (a) We will represent this mapping as a composition of several more elementary mappings (and for other problems too). First consider $S_1 : z \mapsto 1/(z + 1)$. Under such mapping, we have

$$S_1(1) = 0.5 \qquad S_1(i) = 0.5 - 0.5i \qquad S_1(-1) = \infty$$

and

$$S_1(0) = 1 \qquad S_1(-0.5 + 0.5i) = 1 - i \qquad S_1(-1) = \infty.$$

Since S_1 is Möbius, it maps the outer and inner circles to the lines $x = 0.5$ and $x = 1$, respectively. Since it preserves orientation (see Conway Theorem 3.21), the outer and inner disks correspond to the right side of $\mathfrak{Re} z = 0.5$ and $\mathfrak{Re} z = 1$, respectively. Therefore $S_1(\Omega)$ is the vertical strip $\{z \in \mathbb{C} : \mathfrak{Re} z \in (0.5, 1)\}$.

We now define S_2 to be translation by -0.5 followed by rotation by $\pi/2$ so that the new image becomes $\{z \in \mathbb{C} : \mathfrak{Im} z \in (0, 0.5)\}$.

Then we define S_3 to be $z \mapsto e^{2\pi z}$, whose image is the same as $\exp$ acting on $\{z \in \mathbb{C} : \mathfrak{Re} z = (0, \pi)\}$. Therefore, the image after S_3 is the upper half plane.

Finally, we consider the Möbius map mapping the upper half plane to the disk. This is given by $S_4 : z \mapsto (z - i)/(z + i)$. Then, $S_4 \circ S_3 \circ S_2 \circ S_1$ is the map we seek.

(b) We first use $S_1 : z \mapsto 1/z$ which maps the upper half disk to the lower half plane. Next, we define $S_2 : z \mapsto i(z-1)/(z+1)$ so that it maps the lower half plane to the right half plane and then to the full disk. Then $S_2 \circ S_1$ is a map we seek.

Problem 2

Prove that Möbius transformation T maps $\partial\mathbb{D}$ to $\partial\mathbb{D}$ if and only if it is of the form

$$Tz = a\frac{z-b}{1-\overline{b}z}$$

where $b \in \mathbb{C}$ and $|a| = 1$ or of the form $Tz = a/z$ with $|a| = 1$.

Proof. We can WLOG assume $ad - bc \neq 0$ for otherwise T is constant and $T(\partial\mathbb{D})$ is clearly a singleton. Hence we may assume T^{-1} exists.

Note that a Möbius transformation preserves symmetric points:

$$(Tz^*, Tz_1, Tz_2, Tz_3) = (z^*, z_1, z_2, z_3) = \overline{(z, z_1, z_2, z_3)} = \overline{(Tz, Tz_1, Tz_2, Tz_3)}.$$

We have shown in class that z and $1/\overline{z}$ are symmetric across $\partial\mathbb{D}$. In particular, 0 and ∞ are symmetric.
Let $r := T^{-1}(0)$. Since T^{-1} is Möbius just like T, we have $r^* = 1/\overline{r} = T^{-1}(\infty)$. Therefore T must satisfy $T(r) = 0$ and $T(1/\overline{r}) = \infty$.

(1) If $r = 0$, then $T(0) = 0$ and $T(\infty) = \infty$. If $T(z_2) = 1$ then $|z_2| = 1$. Therefore by cross ratio this mapping is $(z, z_2, 0, \infty)$ or $z \mapsto z/z_2 = z_2^* z$, which can indeed be re-written as form

$$Tz = z_2^* \frac{z-0}{1-0z} \qquad \text{where } |z_2^*| = 1.$$

Conversely, if $Tz = a\frac{z-b}{1-\overline{b}z}$ with $b \in \mathbb{C}, |a| = 1$ and 0 is a root, then $b = 0$ and this reduces to $Tz = az$ with $|a| = 1$. This is surely a mapping from $\partial\mathbb{D}$ to $\partial\mathbb{D}$.

Therefore we showed that if $T(0) = 0$ then $T : \partial\mathbb{D} \to \partial\mathbb{D}$ if and only if it is of form

$$Tz = a\frac{z-b}{1-\overline{b}z} \qquad \text{where } b \in \mathbb{C}, |a| = 1.$$

(2) If $r = \infty$, then $T(\infty) = 0$ and $T(0) = \infty$. If $T(z_2) = 1$ then $|z_2| = 1$. Then the mapping corresponds to $(z, z_2, \infty, 0)$ or $z \mapsto z_2/z$.

Conversely, if $T : z \mapsto a/z$ with $|a| = 1$, then if $|z| = 1$ we have $|a/z| = 1$, i.e., $T : \partial\mathbb{D} \to \partial\mathbb{D}$.

Therefore we showed that if $T(\infty) = 0$ then $T : \partial\mathbb{D} \to \partial\mathbb{D}$ if and only if it is of form $Tz = a/z$ with $|a| = 1$.

(3) Finally, if $r \notin \{0, \infty\}$ then $T(1/\overline{r}) = \infty$. The transformation must be given by

$$Tz = a\frac{z-r}{1/\overline{r} - z} = (a\overline{r})\frac{z-r}{1-\overline{r}z}.$$

For $z \in \partial\mathbb{D}$, we have

$$|Tz|^2 = Tz\overline{Tz} = a\overline{r} \cdot \overline{a}r \cdot \frac{(z-r)(\overline{z}-\overline{r})}{(1-\overline{r}z)(1-r\overline{z})} = |a\overline{r}|^2 \cdot \frac{z\overline{z} + r\overline{r} - 2\mathfrak{Re}(\overline{r}z)}{1 + r\overline{r}z\overline{z} - 2\mathfrak{Re}(\overline{r}z)}$$
$$= |a\overline{r}|^2 \cdot \frac{|z|^2 + |r|^2 - 2\mathfrak{Re}(\overline{r}z)}{1 + |rz|^2 - 2\mathfrak{Re}(\overline{r}z)} = |a\overline{r}|^2 \cdot \frac{2 - 2\mathfrak{Re}(\overline{r}z)}{2 - 2\mathfrak{Re}(\overline{r}z)} = |a\overline{r}|^2.$$

That is, $T : \partial\mathbb{D} \to \partial\mathbb{D}$ if and only if $|a\overline{r}| = 1$. Renaming this variable as a, we see that if $T(0), T(\infty) \neq 0$ then $T : \partial\mathbb{D} \to \partial\mathbb{D}$ if and only if it is of the same form as stated in (1).

Combining all three cases, we are done with the proof. $\square$

Problem 3

Assume that $f(z)$ is analytic with $f'(z)$ continuous and that it satisfies $|f(z) - 1| < 1$ in Ω. Prove that

$$\int_\gamma \frac{f'(z)}{f(z)}\, \mathrm{d}z = 0$$

for every closed curve in Ω.

Proof. Since $|f(z) - 1| < 1$ in Ω, $1 < f(z) < 2 < 2\pi$. Therefore $\log(f(z))$ is an analytic on Ω. Since

$$\frac{\mathrm{d}}{\mathrm{d}z}\log(f(z)) = \frac{f'(z)}{f(z)},$$

we see that the integrand admits an analytic primitive. Hence the integral only depends on endpoints, and for a closed interval this means its value is 0. □

Problem 4

Map conformally the exterior of the ellipse
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$
where $a, b > 0$ to the exterior of the unit circle.

Solution. *(Step 1).* We first construct a conformal map from the exterior of an related ellipse (to be specified later) to that of some circle with radius > 1 (i.e., it maps to $\{z \in \mathbb{C} : |z| > r\}$ for some $r > 0$). Assume $a > b$ (otherwise, skip to step 2).

Let $r > 1$. Using the mapping $T : z \mapsto z + 1/z$ and parametrization $re^{i\theta}$ for $\theta \in [0, 2\pi]$,
$$\begin{aligned} T(e^{i\theta}) &= re^{i\theta} + r^{-1}e^{-i\theta} \\ &= r(\cos\theta + i\sin\theta) + r^{-1}(\cos(-\theta) + \sin(-\theta)) \\ &= (r + r^{-1})\cos\theta + (r - r^{-1})\sin\theta \end{aligned}$$
we see that T maps C_r to the ellipse with semiaxes $r + r^{-1}$ and $r - r^{-1}$.

Since $(r + r^{-1})^2 - (r - r^{-1})^2 = 4$, if we let the scaling factor $k > 0$ be such that $k^2(a^2 - b^2) = 4$, then
$$\begin{cases} ka = r + r^{-1} \\ kb = r - r^{-1} \end{cases} \qquad r > 1 \tag{1}$$
has a unique solution. The existence is because of the scaling factor; the uniqueness is because if $r + r^{-1} = s + s^{-1}$ and $r, w > 1$, then
$$(r - s) + (r^{-1} - s^{-1}) = \frac{rs(r - s) - (r - s)}{rs} = \frac{(r - s)(rs - 1)}{rs} = 0,$$
which has no solution unless $r = s$. Denote $r(a, b)$ as the solution satisfying (1).

From above, we see that $z \mapsto 1/z$ maps $\{z \in \mathbb{C} : |z| > r(a, b)\}$ bijectively to the exterior of the ellipse $x^2/(ka)^2 + y^2/(kb)^2 = 1$. (Injectivity follows since $r(a, b) > 1$ and larger circles maps to larger ellipses and surjectivity is clear.)

(Step 2). We construct a mapping $S_1 : z \mapsto kz$ so that the exterior of $x^2/a^2 + y^2/b^2 = 1$ is mapped to the exterior of $x^2/(ka)^2 + y^2/(kb)^2 = 1$.

(Step 3). We construct a conformal mapping from the exterior of $x^2/(ka)^2 + y^2/(kb)^2 = 1$ to $\{z \in \mathbb{C} : |z| > r(a, b)\}$ using the inverse of $z \mapsto z+1/z$ as stated in (1). Call this mapping S_2. Since $z \mapsto z+1/z$ is conformal and bijective (on its domain), by the inverse function theorem, so is its inverse S_2.

(Step 4). We finally construct $S_3 : z \mapsto z/r(a, b)$ so that $\{z \in \mathbb{C} : |z| > r(a, b)\}$ gets mapped to the exterior of our desired unit circle.

Then, $S_3 \circ S_2 \circ S_1$ is the mapping we seek:
$$\begin{aligned} \text{Exterior of } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 &\xrightarrow{S_1} \text{exterior of } \frac{x^2}{(ka)^2} + \frac{y^2}{(kb)^2} = 1 \\ &\xrightarrow{S_2} \{z \in \mathbb{C} : |z| > r(a, b)\} \\ &\xrightarrow{S_3} \{z \in \mathbb{C} : |z| > 1\}. \end{aligned}$$