Beginning of Jan.10, 2022

0.1 Power Series

A **power series** around $a \in \mathbb{C}$ is an infinite series of the form $\sum_{n=0}^{\infty} a_n (z-a)^n$.

Some examples of series:

(1) A boring one that diverges everywhere except at origin: $\sum_{n=0}^{\infty} n! z^n$.

(2) The exponential, the sine, and the cosine functions:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$
$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$
$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots$$

All three converges for all $z \in \mathbb{C}$.

(3) Complex logarithm:

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots$$

which converges for $|z| < 1$ (also for $z = 1$).

(4) $1 + z + z^2 + \ldots = 1/(1-z)$ converges for $|z| < 1$.

Recall a theorem from 425b:

Theorem 0.1.1

For a power series $\sum_{n=0}^{\infty} a_n (z-a)^n$, we define the **radius of convergence** $R \in [0, \infty]$ by

$$R := \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}}.$$

Then:

(1) If $|z-a| < R$ then the series converges absolutely,

(2) If $|z-a| > R$, then the series diverge, and

(3) If $r \in (0, R)$, then the series converges uniformly on the disk

$$D_r(a) := \{z \in \mathbb{C} : |z-a| < r\}.$$

The claims can be easily proven using e.g. the root test.

Proposition 0.1.2

Assume that $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$ have radii of convergence $\geqslant r$. Then the power series $\sum_{n=0}^{\infty} c_n z^n$, where $c_n := \sum_{k=0}^{n} a_n b_{n-k}$ the convolution product, has a radius of convergence $\geqslant r$ as well.

Idea of proof. Assume that $|z| \leqslant r_0 < r$ where r_0 is fixed. Then

$$\sum_{n=0}^{\infty} |c_n||z|^n \leqslant \left(\sum_{n=0}^{\infty} |a_n| r_0^n\right)\left(\sum_{n=0}^{\infty} |b_n| r_0^n\right).$$

□

0.2 Analytic Functions

Let $\Omega \subset \mathbb{C}$ be open. We say a function $f : \Omega \to \mathbb{C}$ is **(complex) differentiable** at $z \in \Omega$ if

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists and is finite. The value $f'(z)$ is called the **(complex) derivative** of f at z. We say $w = \lim_{h\to 0} f(h)$ if for all $\epsilon > 0$, there exists δ such that

$$|h| < \delta \text{ and } h \neq 0 \implies |w - f(h)| < \epsilon.$$

Note that everything resembles what was seen in real analysis, except here we are dealing with complex numbers.

Definition 0.2.1: Analytic Functions

A function $f : \Omega \to \mathbb{C}$ is **analytic** (or **holomorphic**) in Ω if it is differentiable at every $z \in \Omega$.

Remark.

(1) In this course, we use the word “analytic” and “holomorphic” interchangeably.

(2) We don’t assume continuity of f'. A beautiful fact about complex analysis is that if a function is complex differentiable then it is infinitely many times differentiable, i.e., holomorphic.