Sums, Differences, and Products of Analytic Functions

Some basic properties of analytic functions:

(1)   Sums, differences, and products of analytic functions are analytic.

(2)   The quotients, where the denominators do not vanish, are also analytic.

Proposition 0.0.1: Composition of Analytic Functions, Chain Rule

Assume that f, g are analytic in Ω and G respectively, and assume $f(\Omega) \subset G$. Then $g \circ f$ is analytic on Ω and

$$(g \circ f)(z) = g'(f(z))f'(z).$$

Proof. Ideally we would like to use the definition

$$
\begin{aligned}
(g \circ f)'(z) &= \lim_{h \to 0} \frac{(g \circ f)(z+h) - (g \circ f)(z)}{h} \\
&= \lim_{h \to 0} \frac{(g \circ f)(z+h) - (g \circ f)(z)}{f(z+h) - f(z)} \cdot \frac{f(z+h) - f(z)}{h},
\end{aligned}
\tag{*}
$$

but $f(z+h) - f(z)$ could be zero.

Let $z \in \Omega$. It suffices to show that every sequence $\{h_n\} \to 0$ with $h_n \neq 0$ has a subsequence h_{n_k} such that

$$\frac{(g(f(z+h_{n_k})) - g(f(z)))}{h_{n_k}} \to g'(f(z))g'(z),$$

since showing a sequence converges is equivalent to showing that every sequence has a further subsequence.

We have two cases here:

(Case 1) $f(z) \neq f(z + h_n)$ for all n. Then we simply apply (*) and obtain our desired result.

(Case 1.1) $f(z) \neq f(z + h_0)$ for all but finitely many h_n's. We can still apply (*).

(Case 2) $f(z) = f(z + h_n)$ for infinitely many n. WLOG we may assume that this holds for all n. Then

$$\frac{g(f(z+h_n)) - g(f(z))}{h_n} = 0 \qquad \text{for all } n$$

and $g'(f(z))f'(z) = 0$. Then we have $0 = 0$, which still holds.

Note that $f(z+h) - f(z) \to 0$ as $h \to 0$ because f is assumed to be continuous. □

Definition 0.0.2

A function f analytic in all of $\mathbb{C}$ is called **entire**.

Now we provide some examples of analytic/entire functions:

(1) z^n, with $(z^n)' = nz^{n-1}$.

(2) $e^z := e^{x+iy} = e^x(\cos y + i \sin y)$.

Proof. Assuming we know $e^{z_1+z_2} = e^{z_1}e^{z_2}$, we only have to check that

$$\lim_{z \to 0} \frac{e^z - 1}{z} = 1.$$

The proof is left as an exercise in the first problem set — use $\epsilon - \delta$ since this is a complex limit!

(3) $\sin z = (e^{iz} - e^{-iz})/2$ and $\cos z = (e^{iz} + e^{-iz})/2$ are entire.

Facts about periodic functions:

(1) e^z is periodic with period $2\pi i$ (since $e^{x+2\pi i} = e^x e^{2\pi i} = e^x(\cos(2\pi) + i\sin(2\pi)) = e^x$).

(2) $\sin z, \cos z$ are periodic with periods 2π.

Theorem 0.0.3: Inverse Function Theorem

Let $\Omega, G \subset \mathbb{C}$ be open. Assume that $f : \Omega \to \mathbb{C}$ and $g : G \to \mathbb{C}$ are continuous. Also assume that $f(\Omega) \subset G$ and $g(f(z)) = z$ for all $z \in \Omega$ (so that g is the "inverse" of f). If g is differentiable at $z \in C$ and $g'(f(z)) \neq 0$, then f is differentiable at z with

$$f'(z) = \frac{1}{g'(f(z))}.$$

Note again that this is about complex variables which is different from the real-valued case.

Proof. Let $h \neq 0$ be small. Note that

$$1 = \frac{h}{h} = \frac{g(f(z+h)) - g(f(z))}{h} = \frac{g(f(z+h)) - g(f(z))}{f(z+h) - g(z)} \cdot \frac{f(z+h) - f(z)}{h}.$$

Since f is injective (as we assumed $g(f(z)) = z$ for all z which is impossible if f is not injective), we have $f(z+h) - f(z) \neq 0$. Also, by continuity

$$\lim_{h \to 0} (f(z+h) - f(z)) = 0,$$

so

$$\lim_{h \to 0} \frac{g(f(z+h)) - g(f(z))}{f(z+h) - f(z)} = g'(f(z)),$$

and of course the second term has to converge to $1/g'(f(z))$. $\qquad\square$

0.1   Complex Logarithm

The problem. e^z is *not* bijective (recall it is periodic).