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**Definition 0.0.1**

Let $f : \Omega \to \mathbb{C}$ where $\Omega$ is open. Let $f$ be continuous and such that

$$z = \exp(f(z)) \qquad \text{for } z \in \Omega.$$

Then $f$ is called a **branch of the logarithm**.

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Since $e^{z+2\pi i} = e^z$ for all $z \in \mathbb{C}$, we have the following result:

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**Proposition 0.0.2**

If $f, g$ are two branches of the logarithm function on $\Omega \subset \mathbb{C}$, then

$$f(z) = g(z) + 2\pi k i$$

where $k \in \mathbb{Z}$ is fixed.

Conversely, if $f$ on $\Omega$ is a branch of $\log$, then so is $f(z) + 2\pi k i$ where $k \in \mathbb{Z}$.

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From this definition, different $\Omega$'s will give to different $\log$'s.

We will use the following branch of the $\log$ called the **principle branch of the logarithm**:

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**Definition 0.0.3: Principle Branch**

Let $\Omega := \mathbb{C} \setminus (-\infty, 0]$, and we represent $z \in \Omega$ as

$$z = |z|e^{i\theta} \qquad \text{(polar representation)}$$

where $-\pi < \theta < \pi$. Note that $\theta$ is a continuous function of $z \in \mathbb{C}$. We then let

$$f(re^{i\theta}) = \log r + i\theta \qquad \text{for } r > 0, -\pi < \theta < \pi.$$

This is called the **principle branch**.

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This indeed makes sense, as

$$\exp(f(re^{i\theta})) = \exp(\log r + i\theta) = re^{i\theta},$$

so indeed $\exp(f(z)) = z$ for all defined $z$.

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**Proposition 0.0.4**

Every branch of the $\log$ is analytic in $\Omega$ and the derivative is $1/z$.

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# Chapter 1

# Cauchy-Riemann Equations

A function of $x, y$ can be considered as a function of $z$ and $\overline{z}$ where

$$x = \frac{z + \overline{z}}{2} \qquad \text{and } y = \frac{z - \overline{z}}{2i}.$$

A function $g(z, \overline{z})$ is analytic if $\dfrac{\partial}{\partial \overline{z}} g(z, \overline{z}) = 0$. In other words, analytic functions are the ones that only depend on $z$ but not $\overline{z}$.

Assume that

$$f'(z) = \lim_{h \to 0} \frac{f(z + h) - f(z)}{h}$$

exists. If we let $u = \mathfrak{Re} f$ and $v = \mathfrak{Im} f$, then $f = u + iv$. We also let $z = x + iy$ where $x, y \in \mathbb{R}$. (We will use these notations frequently.)

Consider $h \to 0$ along the real line. Then

$$\lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{f(z + h) - f(z)}{h} = \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{u(x + h, y) - u(x, y)}{h} + i \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{v(x + h, y) - v(x, y)}{h}$$

$$= \frac{\partial u}{\partial x}(x, y) + i \frac{\partial v}{\partial x}(x, y).$$

Therefore, if $f$ is analytic, $\partial u / \partial x$ and $\partial v / \partial x$ exist at $z$, with

$$f' = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}, \tag{1}$$

where we have implicitly assumed that $y$ is held constant.

Now, we let $h \to 0$ along the imaginary values. That is, we switch $h$ to $ih$:

$$\lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{f(z + ih) - f(z)}{ih} = \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{u(x, y + ih) - u(x, y)}{ih} + i \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{v(x, y + ih) - v(x, y)}{ih}$$

$$= \frac{\partial v}{\partial y}(x, y) - i \frac{\partial u}{\partial y}.$$

Therefore, assuming $f'$ exists, we see $\partial v / \partial y$ and $\partial v / \partial y$ both exist at $z = x + iy$ with

$$f' = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}. \tag{2}$$

Comparing (1) with (2), we obtain the **Cauchy-Riemann Equations**:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{CR}$$

The converse also holds. Now we put everything into a theorem:

> **Theorem 1.0.1: Analytic Functions & the Cauchy-Riemann Equation**
>
> Let $u, v : \Omega \to \mathbb{R}$ and let $f = u + iv$, a complex function $\Omega \to \mathbb{C}$.
> As mentioned above, if $f$ is analytic in $\Omega$, then $u, v$ satisfy (CR). Conversely, if $u, v \in C^1(\Omega)$ satisfy (CR), then
> $f = u + iv$ is analytic in $\Omega$.

*Proof.* It remains to show that (CR) $\Rightarrow$ ($f$ is analytic), so assume (CR) holds with $u, v \in C^1$. Let $h, k \in \mathbb{R}$, and define

$$\varphi(h, k) := u(x + h, y + k) - u(x, y) - hu_x(x, y) - ku_y(x, y)$$
$$= u(x + h, y + k) - u(x, y + k) - hu_x(x, y)$$
$$+ u(x, y + k) - u(x, y) - ku_y(x, y).$$

(We perturb $x$ slightly in the first three terms and perturb $y$ slightly in the last three.) Using MVT (since $u, v \in C^1$), there exist $h_1 \in (0, h)$ and $k_1 \in (0, k)$ such that

$$\varphi(h, k) = hu_x(x + h_1, y + k) - hu_x(x, y)$$
$$+ ku_y(x, y + h_1) - ku_y(x, y).$$

Taking limits as $h + ik \to 0$, we obtain that $\lim_{h+ik \to 0} \varphi(h, k)/(h + ik) = 0$: the first two terms when divided by $h + ik$ become

$$\frac{h}{h + ik}(u_x(x + h, y + k) - u_x(x, y))$$

where $|h/(h + ik)| \leq 1$ and the second term $\to 0$. Similar argument can be made for the last two terms. To sum up,

$$u(x + h, v + k) - u(x, y) = u_x(x, y)h + u_y(x, y)k + \varphi(h, k)$$

and similarly

$$v(x + h, v + k) - v(x, y) = v_x(x, y)h + v_y(x, y)k + \psi(h, k)$$

where

$$\lim_{h+ik \to 0} \frac{\varphi(h, k)}{h + ik} = \lim_{h+ik \to 0} \frac{\psi(h, k)}{h + ik} = 0.$$

We know that $u, v \in C^1$. Therefore,

$$\lim_{h+ik \to 0} \frac{f(z + h + ik) - f(z)}{h + ik} = u_x(z) + iv_x(z) + \lim_{h+ik \to 0} \frac{\varphi(h, k) + i\psi(h, k)}{h + ik} \tag{$\Delta$}$$

where we used (CR) and the identity

$$u_x h - v_x k + i(v_x h + u_x k) = (u_x + iv_x)(h + ik).$$

Since the last term in ($\Delta$) $\to 0$, $f'$ exists and equals $(u_x + iv_x)(z)$, as claimed. $\qquad \square$