Beginning of Feb.7, 2021

Proof. Let $A : \{z \in \mathbb{D} : \mathfrak{Im} z \neq \mathfrak{Im} z_i, \mathfrak{Re} z \neq \mathfrak{Re} z_i\}$, i.e., the collection of "good" points whose corresponding vertical and horizontal lines contain no singular points. We choose $z_0 \in A$. For $z \in \Omega$, choose any vertical-horizontal-vertical path σ_1 avoiding $z_1, \ldots, z_n$.

Note that the definition $F(z) := \int_{\sigma_1} f(z)\, \mathrm{d}z$ is well-defined and independent of choice of the horizontal part of σ_1. This is because of the "generalized" Cauchy's rectangle theorem which states that the integral around a rectangle, even if there are bad points inside, is 0.

Because of this, the y-derivative of F depends only on the last vertical segment, that is,

$$\frac{\partial F}{\partial y} = if.$$

Similarly, we define σ_2 to be a horizontal-vertical-horizontal path, also avoiding $z_1, \ldots, z_n$, and by the same token

$$\frac{\partial F}{\partial x} = f.$$

(We can easily check that with the same endpoinds, integral over σ_1 and σ_2 are indeed the same by using the rectangle theorem twice, so the resulting capital function is indeed F). Then we have

$$\frac{\partial F}{\partial x} = -i\frac{\partial F}{\partial y}$$

so F is analytic. It remains to notice that

$$F' = \frac{\partial \mathfrak{Re} F}{\partial x} + i\frac{\partial \mathfrak{Im} F}{\partial x} = \frac{\partial \mathfrak{Re} F}{\partial x} - i\frac{\partial \mathfrak{Re} F}{\partial y} = \mathfrak{Re} f - i(-\mathfrak{Im} f) = \mathfrak{Re} f + i\mathfrak{Im} f = f.$$ □

Upshot. If F is analytic and $\frac{\partial F}{\partial x} = f, \frac{\partial F}{\partial y} = if$, then $F'(z) = f(z)$.

0.1 Index / Winding Number

To define the index of a curve around a point, we need the next lemma:

Lemma 0.1.0.1

Let $a \in \mathbb{C}$ and let γ be a piecewise smooth *closed* curve that does *not* pass through a. Then

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{z-a}\,\mathrm{d}z \in \mathbb{Z}.$$

Recall that γ is oriented; that is, reversing the orientation flips the sign of the number above and therefore the index. We assume γ is piecewise smooth.

Definition 0.1.1: Index

For $a \in \mathbb{C}$ not on the curve $\{\gamma\}$ (the image of $\gamma : [\alpha, \beta] \to \mathbb{C}$), we define the **index / winding number** of γ with respect to a as

$$n(\gamma, a) := \frac{1}{2\pi i}\int_\gamma \frac{1}{z-a}\,\mathrm{d}z.$$

Motivation for winding number.

If $\log$ exists on some γ (not necessarily closed), then

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{z-a}\,\mathrm{d}z = \frac{1}{2\pi i}(\log(z_2-a)-\log(z_1-a)) = \frac{1}{2\pi i}\log\frac{|z_2-a|}{|z_1-a|}+\frac{1}{2\pi}(\arg z_2-\arg z_1).$$

As γ approaches a closed curve, $\arg z_2 - \arg z_1$ approaches $\pm 2\pi$, and the first quotient $\to 0$. That is,

Proof of Lemma. Let $z(t)$, $t \in [\alpha, \beta]$, be a parametrization of γ. Let

$$h(t) := \int_\alpha^t \frac{z'(s)}{z(t)-a}\,\mathrm{d}s,$$

which is well-defined since $z(t) \neq a$ for all t.

Idea: we expect $h(t) = \log(z(t)-a)$ but we have some technical difficulties. Thus we consider the exponential.

We have

$$(e^{-h(t)}(z(t)-a))$$

□