Beginning of March 2, 2022

0.1 Local Mapping Properties

There are a number of ways to count zeroes in regions.

Theorem 0.1.1

Let $z_1, z_2, \ldots$ be all the zeros of f that is not identically zero in $D_r(a)$. Let γ be a closed curve in D which does not pass through any zeros. Then

$$\sum n(\gamma, z_i) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, \mathrm{d}z.$$

This implies that almost all zeros have zero index.

Since analytic functions have discrete zeros, if f (nonzero) is analytic in a connected Ω and $K \subset \Omega$ is compact, then f can only have finitely many zeros in K. We will use his fact in the proof.

Proof. By reducing the radius we may assume that there are only finitely many zeros, $z_1, \ldots, z_n$ of f, repeated according to their multiplicities. (Use the compact argument above.)

Then there exists a nonzero g in D such that

$$f(z) = (z - z_1)(z - z_2)\ldots(z - z_n)g(z).$$

We define the *logarithmic derivative* by applying the Leibniz rule:

$$f'(z) = \sum_{i=1}^{n} g(z) \prod_{j \neq i} (z - z_j) \implies \frac{f'(z)}{f(z)} = \frac{1}{z - z_1} + \ldots + \frac{1}{z - z_n} + \frac{g'(z)}{g(z)}.$$

Integrating over γ, we have

$$\int_\gamma \frac{f'(z)}{f(z)}\, \mathrm{d}z = 2\pi i \sum_{i=1}^{n} n(\gamma, z_n) + 0 \qquad \square$$

Using chance of variable $w = f(z)$, we have

$$\int_\gamma \frac{f'(z)}{f(z)} = \int_{f \circ \gamma} \frac{\mathrm{d}w}{w}.$$

Then the number of zeros in γ is simply the index of 0 in $f \circ \gamma$. That is,

Corollary 0.1.2

Under the assumptions of the previous theorem, we have

$$\sum n(\gamma, z_j) = n(\Gamma, 0),$$

where $\Gamma = f \circ \gamma$. In particular, if γ is a simple closed curve which does not intersect itself, then the total number of zeros inside γ (with multiplicities counted repeatedly) equals the index of $f \circ \gamma$ around 0.

This can be applied to $f(z) - a$ for any $a \in \mathbb{C}$. We obtain

$$\sum_j n(r, z_j(a)) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z) - a} \, \mathrm{d}a$$

where $z_j(a)$ are the zeros of $f - a$. After changing variable we see that the RHS is also the index of $f \circ \gamma$ around a.

Upshot. The winding number does not change if we perturb a (i.e., if Δa is small).

Theorem 0.1.3

Assume that f is analytic at z_0 and $f(z) - w_0$ has a zero of order n at z_0. Then for $\epsilon > 0$ sufficiently small, there exists $\delta > 0$ such that

$$f(z) - a \text{ has } n \text{ distinct simple zeros} \qquad \text{for all } a \in D_\delta(w_0) \backslash \{w_0\}.$$