Beginning of March 30, 2022

Theorem 0.0.1: Residue Theorem

Assume that f is analytic except on a discrete set $A = \{a_1, a_2, \ldots\} \subset \Omega$ and that f is analytic in $\Omega \backslash A$. Then

$$\int_\gamma f \, \mathrm{d}z = 2\pi i \sum_j n(\gamma, a_j) \operatorname{Res}(f, a_j)$$

for every $\gamma \in C_0$ not intersecting A and $\gamma \sim 0$. Since A is discrete, we have $n(\gamma, a_j) = 0$ for all but finitely many.

Proof. WLOG assume $A = \{a_1, \ldots, a_n\}$ is finite (for we can always throw away those with index 0). Let $\gamma_1, \ldots, \gamma_n$ be sufficiently small circles so that the disks are separated. Also assume that these disks are entirely in Ω. Then

$$g(z) := f - \sum_{j=1}^{n} \frac{\operatorname{Res}(f, a_j)}{z - a_j}$$

is periodic. So

$$\int_\gamma f - \sum_{j=1}^{n} \frac{\operatorname{Res}(f, a_j)}{z_j} \, \mathrm{d}z = 2\pi i \sum_j \operatorname{Res}(f, a_j) n(\gamma, a_j).$$

□

Computing Residues at Poles

If f has a pole of order n, then

$$f(z) = \frac{g(z)}{(z-a)^n}$$

where g is analytic and $g(0) \neq 0$. By Taylor's theorem,

$$f(z) = ((g(0) + (z-a)g'(p) + \ldots + (z-a)^{n-1} \cdot \frac{g^{n-1}(a)}{(n-1)!} + (z-a)^n h(z)$$

where $h(z)$ is also analytic.