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Chapter 1

Introductions

1.1 Power Series

Beginning of Jan.10, 2022

A **power series** around $a \in \mathbb{C}$ is an infinite series of the form $\sum_{n=0}^{\infty} a_n (z-a)^n$.

Some examples of series:

(1) A boring one that diverges everywhere except at origin: $\sum_{n=0}^{\infty} n! z^n$.

(2) The exponential, the sine, and the cosine functions:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$
$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$
$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots$$

All three converges for all $z \in \mathbb{C}$.

(3) Complex logarithm:

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots$$

which converges for $|z| < 1$ (also for $z = 1$).

(4) $1 + z + z^2 + \ldots = 1/(1-z)$ converges for $|z| < 1$.

Recall a theorem from 425b:

Theorem 1.1.1

For a power series $\sum_{n=0}^{\infty} a_n (z-a)^n$, we define the **radius of convergence** $R \in [0, \infty]$ by

$$R := \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}}.$$

Then:

(1) If $|z - a| < R$ then the series converges absolutely,

(2) If $|z - a| > R$, then the series diverge, and

(3) If $r \in (0, R)$, then the series converges uniformly on the disk

$$D_r(a) := \{z \in \mathbb{C} : |z - a| < r\}.$$

The claims can be easily proven using e.g. the root test.

Proposition 1.1.2

Assume that $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$ have radii of convergence $\geqslant r$. Then the power series $\sum_{n=0}^{\infty} c_n z^n$, where $c_n := \sum_{k=0}^{n} a_n b_{n-k}$ the convolution product, has a radius of convergence $\geqslant r$ as well.

Idea of proof. Assume that $|z| \leqslant r_0 < r$ where r_0 is fixed. Then

$$\sum_{n=0}^{\infty} |c_n||z|^n \leqslant \left(\sum_{n=0}^{\infty} |a_n| r_0^n\right)\left(\sum_{n=0}^{\infty} |b_n| r_0^n\right).$$

□

1.2 Analytic Functions

Let $\Omega \subset \mathbb{C}$ be open. We say a function $f : \Omega \to \mathbb{C}$ is **(complex) differentiable** at $z \in \Omega$ if

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists and is finite. The value $f'(z)$ is called the **(complex) derivative** of f at z. We say $w = \lim_{h \to 0} f(h)$ if for all $\epsilon > 0$, there exists δ such that

$$|h| < \delta \text{ and } h \neq 0 \implies |w - f(h)| < \epsilon.$$

Note that everything resembles what was seen in real analysis, except here we are dealing with complex numbers.

Definition 1.2.1: Analytic Functions

A function $f : \Omega \to \mathbb{C}$ is **analytic** (or **holomorphic**) in Ω if it is differentiable at every $z \in \Omega$.

Remark.

(1) In this course, we use the word "analytic" and "holomorphic" interchangeably.

(2) We don't assume continuity of f'. A beautiful fact about complex analysis is that if a function is complex differentiable then it is infinitely many times differentiable, i.e., holomorphic.

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Sums, Differences, and Products of Analytic Functions

Some basic properties of analytic functions:

(1) Sums, differences, and products of analytic functions are analytic.

(2) The quotients, where the denominators do not vanish, are also analytic.

Proposition 1.2.2: Composition of Analytic Functions, Chain Rule

Assume that f, g are analytic in Ω and G respectively, and assume $f(\Omega) \subset G$. Then $g \circ f$ is analytic on Ω and

$$(g \circ f)(z) = g'(f(z))f'(z).$$

Proof. Ideally we would like to use the definition

$$\begin{aligned}(g \circ f)'(z) &= \lim_{h\to 0} \frac{(g \circ f)(z+h) - (g \circ f)(z)}{h} \\ &= \lim_{h\to 0} \frac{(g \circ f)(z+h) - (g \circ f)(z)}{f(z+h) - f(z)} \cdot \frac{f(z+h) - f(z)}{h}, \qquad (*)\end{aligned}$$

but $f(z+h) - f(z)$ could be zero.

Let $z \in \Omega$. It suffices to show that every sequence $\{h_n\} \to 0$ with $h_n \neq 0$ has a subsequence h_{n_k} such that

$$\frac{(g(f(z+h_{n_k})) - g(f(z)))}{h_{n_k}} \to g'(f(z))g'(z),$$

since showing a sequence converges is equivalent to showing that every sequence has a further subsequence. We have two cases here:

(Case 1) $f(z) \neq f(z+h_n)$ for all n. Then we simply apply (*) and obtain our desired result.

(Case 1.1) $f(z) \neq f(z+h_0)$ for all but finitely many h_n's. We can still apply (*).

(Case 2) $f(z) = f(z+h_n)$ for infinitely many n. WLOG we may assume that this holds for all n. Then

$$\frac{g(f(z+h_n)) - g(f(z))}{h_n} = 0 \qquad \text{for all } n$$

and $g'(f(z))f'(z) = 0$. Then we have $0 = 0$, which still holds.

Note that $f(z+h) - f(z) \to 0$ as $h \to 0$ because f is assumed to be continuous. □

Definition 1.2.3

A function f analytic in all of $\mathbb{C}$ is called **entire**.

Now we provide some examples of analytic/entire functions:

(1) z^n, with $(z^n)' = nz^{n-1}$.

(2) $e^z := e^{x+iy} = e^x(\cos y + i \sin y)$.

Proof. Assuming we know $e^{z_1+z_2} = e^{z_1}e^{z_2}$, we only have to check that

$$\lim_{z\to 0}\frac{e^z-1}{z} = 1.$$

The proof is left as an exercise in the first problem set — use $\epsilon - \delta$ since this is a complex limit!

(3) $\sin z = (e^{iz} - e^{-iz})/2$ and $\cos z = (e^{iz} + e^{-iz})/2$ are entire.

Facts about periodic functions:

(1) e^z is periodic with period $2\pi i$ (since $e^{x+2\pi i} = e^x e^{2\pi i} = e^x(\cos(2\pi) + i\sin(2\pi)) = e^x$).

(2) $\sin z, \cos z$ are periodic with periods 2π.

Theorem 1.2.4: Inverse Function Theorem

Let $\Omega, G \subset \mathbb{C}$ be open. Assume that $f : \Omega \to \mathbb{C}$ and $g : G \to \mathbb{C}$ are continuous. Also assume that $f(\Omega) \subset G$ and $g(f(z)) = z$ for all $z \in \Omega$ (so that g is the "inverse" of f). If g is differentiable at $z \in C$ and $g'(f(z)) \neq 0$, then f is differentiable at z with

$$f'(z) = \frac{1}{g'(f(z))}.$$

Note again that this is about complex variables which is different from the real-valued case.

Proof. Let $h \neq 0$ be small. Note that

$$1 = \frac{h}{h} = \frac{g(f(z+h)) - g(f(z))}{h} = \frac{g(f(z+h)) - g(f(z))}{f(z+h) - g(z)} \cdot \frac{f(z+h) - f(z)}{h}.$$

Since f is injective (as we assumed $g(f(z)) = z$ for all z which is impossible if f is not injective), we have $f(z+h) - f(z) \neq 0$. Also, by continuity

$$\lim_{h\to 0}(f(z+h) - f(z)) = 0,$$

so

$$\lim_{h\to 0}\frac{g(f(z+h)) - g(f(z))}{f(z+h) - f(z)} = g'(f(z)),$$

and of course the second term has to converge to $1/g'(f(z))$. □

1.3 Complex Logarithm

The problem. e^z is *not* bijective (recall it is periodic).

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Definition 1.3.1

Let $f : \Omega \to \mathbb{C}$ where Ω is open. Let f be continuous and such that

$$z = \exp(f(z)) \qquad \text{for } z \in \Omega.$$

Then f is called a **branch of the logarithm**.

Since $e^{z+2\pi i} = e^z$ for all $z \in \mathbb{C}$, we have the following result:

Proposition 1.3.2

If f, g are two branches of the logarithm function on $\Omega \subset \mathbb{C}$, then

$$f(z) = g(z) + 2\pi k i$$

where $k \in \mathbb{Z}$ is fixed.

Conversely, if f on Ω is a branch of $\log$, then so is $f(z) + 2\pi k i$ where $k \in \mathbb{Z}$.

From this definition, different Ω's will give to different $\log$'s.

We will use the following branch of the $\log$ called the **principle branch of the logarithm**:

Definition 1.3.3: Principle Branch

Let $\Omega := \mathbb{C} \setminus (-\infty, 0]$, and we represent $z \in \Omega$ as

$$z = |z|e^{i\theta} \qquad \text{(polar representation)}$$

where $-\pi < \theta < \pi$. Note that θ is a continuous function of $z \in \mathbb{C}$. We then let

$$f(re^{i\theta}) = \log r + i\theta \qquad \text{for } r > 0, -\pi < \theta < \pi.$$

This is called the **principle branch**.

This indeed makes sense, as

$$\exp(f(re^{i\theta})) = \exp(\log r + i\theta) = re^{i\theta},$$

so indeed $\exp(f(z)) = z$ for all defined z.

Proposition 1.3.4

Every branch of the $\log$ is analytic in Ω and the derivative is $1/z$.

Chapter 2

Cauchy-Riemann Equations

2.1 The Cauchy-Riemann Equations

A function of x, y can be considered as a function of z and $\overline{z}$ where

$$x = \frac{z + \overline{z}}{2} \qquad \text{and } y = \frac{z - \overline{z}}{2i}.$$

A function $g(z, \overline{z})$ is analytic if $\frac{\partial}{\partial \overline{z}} g(z, \overline{z}) = 0$. In other words, analytic functions are the ones that only depend on z but not $\overline{z}$.

Assume that

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists. If we let $u = \mathfrak{Re} f$ and $v = \mathfrak{Im} f$, then $f = u + iv$. We also let $z = x + iy$ where $x, y \in \mathbb{R}$. (We will use these notations frequently.)

Consider $h \to 0$ along the real line. Then

$$\begin{aligned} \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{f(z+h) - f(z)}{h} &= \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{u(x+h, y) - u(x, y)}{h} + i \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{v(x+h, y) - v(x, y)}{h} \\ &= \frac{\partial u}{\partial x}(x, y) + i \frac{\partial v}{\partial x}(x, y). \end{aligned}$$

Therefore, if f is analytic, $\partial u / \partial x$ and $\partial v / \partial x$ exist at z, with

$$f' = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}, \tag{1}$$

where we have implicitly assumed that y is held constant.

Now, we let $h \to 0$ along the imaginary values. That is, we switch h to ih:

$$\begin{aligned} \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{f(z+ih) - f(z)}{ih} &= \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{u(x, y+ih) - u(x, y)}{ih} + i \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{v(x, y+ih) - v(x, y)}{ih} \\ &= \frac{\partial v}{\partial y}(x, y) - i \frac{\partial u}{\partial y}. \end{aligned}$$

Therefore, assuming f' exists, we see $\partial v / \partial y$ and $\partial v / \partial y$ both exist at $z = x + iy$ with

$$f' = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}. \tag{2}$$

Comparing (1) with (2), we obtain the **Cauchy-Riemann Equations**:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{CR}$$

The converse also holds. Now we put everything into a theorem:

Theorem 2.1.1: Analytic Functions & the Cauchy-Riemann Equation

Let $u, v : \Omega \to \mathbb{R}$ and let $f = u + iv$, a complex function $\Omega \to \mathbb{C}$.
As mentioned above, if f is analytic in Ω, then u, v satisfy (CR). Conversely, if $u, v \in C^1(\Omega)$ satisfy (CR), then $f = u + iv$ is analytic in Ω.

Proof. It remains to show that (CR) $\Rightarrow$ (f is analytic), so assume (CR) holds with $u, v \in C^1$. Let $h, k \in \mathbb{R}$, and define

$$\begin{aligned}\varphi(h,k) &:= u(x+h, y+k) - u(x,y) - hu_x(x,y) - ku_y(x,y) \\ &= u(x+h,y+k) - u(x,y+k) - hu_x(x,y) \\ &\quad + u(x,y+k) - u(x,y) - ku_y(x,y).\end{aligned}$$

(We perturb x slightly in the first three terms and perturb y slightly in the last three.) Using MVT (since $u, v \in C^1$), there exist $h_1 \in (0,h)$ and $k_1 \in (0,k)$ such that

$$\begin{aligned}\varphi(h,k) &= hu_x(x+h_1, y+k) - hu_x(x,y) \\ &\quad + ku_y(x, y+h_1) - ku_y(x,y).\end{aligned}$$

Taking limits as $h + ik \to 0$, we obtain that $\lim_{h+ik\to 0} \varphi(h,k)/(h+ik) = 0$: the first two terms when divided by $h + ik$ become

$$\frac{h}{h+ik}(u_x(x+h,y+k) - u_x(x,y))$$

where $|h/(h+ik)| \leqslant 1$ and the second term $\to 0$. Similar argument can be made for the last two terms.
To sum up,

$$u(x+h, v+k) - u(x,y) = u_x(x,y)h + u_y(x,y)k + \varphi(h,k)$$

and similarly

$$v(x+h, v+k) - v(x,y) = v_x(x,y)h + v_y(x,y)k + \psi(h,k)$$

where

$$\lim_{h+ik\to 0} \frac{\varphi(h,k)}{h+ik} = \lim_{h+ik\to 0} \frac{\psi(h,k)}{h+ik} = 0.$$

We know that $u, v \in C^1$. Therefore,

$$\lim_{h+ik\to 0} \frac{f(z+h+ik) - f(z)}{h+ik} = u_x(z) + iv_x(z) + \lim_{h+ik\to 0} \frac{\varphi(h,k) + i\psi(h,k)}{h+ik} \tag{Δ}$$

where we used (CR) and the identity

$$u_x h - v_x k + i(v_x h + u_x k) = (u_x + iv_x)(h+ik).$$

Since the last term in (Δ) $\to 0$, f' exists and equals $(u_x + iv_x)(z)$, as claimed. $\square$

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2.2 Harmonic Functions

Now we suppose that $f = u + iv$ be analytic in Ω and further assume that $u, v \in C^2(\Omega)$. By (CR),

$$u_{x,x} = (v_y)_x = (v_x)_y = (-u_y)_y = -u_{y,y}.$$

From this we have

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{H}$$

From this we say functions like u that satisfy (H) are **harmonic**.

If f is analytic, then $-if = v - iu$ is analytic, so v is harmonic. We can also check this directly:

$$v_{x,x} = -(u_y)_x = -(u_x)_y = -v_{y,y} \implies \frac{\partial v^2}{\partial x^2} + \frac{\partial v^2}{\partial y^2} = 0.$$

If u, v are harmonic in Ω and if $f = u + iv$ is analytic in Ω, then we say v is **conjugate harmonic** to u. For example, if v is conjugate harmonic to u, then $-u$ is conjugate harmonic to v.

Example 2.2.1. $e^{\sin y}$ is conjugate harmonic to $e^x \cos y$ because

$$e^x \cos y + ie^x \sin y = e^z.$$

Our next question of interest: given a harmonic function, does there exist a conjugate harmonic function? The answer is no.
For example, let $u = \log\sqrt{x^2 + y^2}$. We claim that there does *not* exist a harmonic conjugate in $\mathbb{C} \setminus \{0\}$, for if there were, $\arg(x + iy)$ fails to be harmonic.

Theorem 2.2.2

Let $\Omega := D_r(z_0)$ (disk) where $r > 0$ and $z_0 \in \mathbb{C}$. Let u be harmonic in Ω. Then u has a harmonic conjugate.
In other words, harmonic conjugate to a given harmonic function always exists locally.

Proof. WLOF assume z_0 is the origin so $\Omega = D_r := D_r(0)$. Suppose first that v, a harmonic conjugate, exists. We will derive an explicit formula for it and then prove that this actually works.
Since (CR) states $v_y = u_x$, we have

$$v(x, y) = \int_0^y u_x(x, t)\ \mathrm{d}t + \varphi(x) =: \int_0^y u_x(x, t)\ \mathrm{d}t + v(x, 0) \tag{*}$$

where $\varphi(x) = v(x, 0)$. We determine φ from the second (CR) $v_x = -u_y$. From (*) we get

$$\begin{aligned} -u_y(x, y) = v_x &= \int_0^y u_{x,x}(x, t)\ \mathrm{d}t + \varphi'(x) \\ &= -\int_0^y u_{y,y}(x, t)\ \mathrm{d}t + \varphi'(x) \\ &= -u_y(x, y) + u_y(x, 0) + \varphi'(x). \end{aligned}$$

Thus $\varphi'(x) = -u_y(x,0)$. Integrating gives

$$\varphi(x) = \varphi(0) + \int_0^x \varphi(t)\ \mathrm{d}t = \varphi(0) - \int_0^x u_y(t,0)\ \mathrm{d}t.$$

Note that $\varphi(0)$ can be any constant and harmonic conjugates are indeed constant indeterminant. Thus for convenience we let

$$\varphi(x) := -\int_0^x u_y(t,0)\ \mathrm{d}t.$$

Therefore, v the harmonic conjugate, *if it exists*, must be given by

$$v = \int_0^y u_x(x,t)\ \mathrm{d}t - \int_0^x u_y(t,0)\ \mathrm{d}t.$$

It remains to check that this is indeed the harmonic conjugate:

$$\begin{aligned} v_x &= \int_0^y u_{x,x}(x,t)\ \mathrm{d}t - u_y(x,0) \\ &= -\int_0^y u_{y,y}(x,t)\ \mathrm{d}t - u_y(x,0) \\ &= -u_y(x,y) + u_y(x,0) - u_y(x_0) = -u_y, \end{aligned}$$

and of course $v_y = u_x$. □

Remark. In this remark, we had a "line integral" that only went horizontally and vertically. In more general cases we will need to use actual line integrals.

One Application of Cauchy-Riemann

Theorem 2.2.3

Let f be analytic in Ω. If any of the following is true, then f must be constant:

(1) f' is constantly zero;

(2) f maps to a line; and

(3) f maps to a circle.

Proof.

(1) Assume $f' = 0$. Since $f' = u_x + iv_x$ we must have $u_x = 0$ and $v_x = 0$. By (CR), $u_y = v_y = 0$. Thus f must be constant.

(2) Multiplying everything by an appropriate constant $e^{i\theta}$ and then translating by another constant, we can assume that the line is $i\mathbb{R}$. Therefore $u = \mathfrak{Re} f = 0$. but then $f' = u_x + iv_x = u_x - iu_y$ by (CR). But $u_x = u_y = 0$. By (a), f is constant.

(3) WLOG we can assume that the circle is centered at the origin with $x^2 + y^2 = a$. Then $u^2 + v^2 = a$, and taking

derivative implies $uu_x + vv_x = 0$ and $uu_y + vv_y = 0$. By (CR),

$$uu_x + vv_x = 0 \qquad \text{and } - uv_x + vu_x = 0$$

This implies $u_x = v_x = 0$ except at points where the determinant is zero, but the determinant is $u^2 + v^2 = a$, so zero means $u = v = 0$ in Ω. Otherwise, $u_x = v_x = 0$ and we can again apply (a).

□

Beginning of Jan.21, 2022

2.3 Analytic Functions as Mappings

We'll prove that analytic f preserves angles at points z_0 where $f'(z_0) \neq 0$. First, some definitions.

Definition 2.3.1: Path

A **path** in $\Omega \subset \mathbb{C}$ is a continuous function $\gamma : [a, b] \to \Omega$ where $-\infty < a < b < \infty$. If γ' exists for every $t \in [a, b]$ and γ' is continuous, we call γ a $\boldsymbol{C^1}$ **path**. A path is **piecewise** $\boldsymbol{C^1}$ if there exists a partition $a = t_0 < t_1 < \ldots < t_n = b$ such that γ is C^1 on each $[t_i - 1, t_i]$.

Definition 2.3.2: Angles

Let γ_1, γ_2 be two *smooth* curves such that $\gamma_1(t_1) = \gamma_2(t_2) = z_0$. The **angle** between γ_1 and γ_2 at z_0 is defined to be as

$$\arg \gamma_2'(t_2) - \arg \gamma_1'(t_1) \in \mathbb{Z}/2\pi\mathbb{Z}.$$

If we assume that $\gamma : \Omega \to \mathbb{C}$ is smooth and $f : \Omega \to \mathbb{C}$ is analytic. Then

$$\tilde{\gamma} := f \circ \gamma$$

is smooth and

$$\tilde{\gamma}'(t) = f'(\gamma(t))\gamma'(t). \tag{*}$$

To prove this, either use differnerence quotients and chain rule or prove by decomposing $\gamma = \gamma_1 + i\gamma_2$, $f = u + iv$.

From (*), we get that

$$\arg \tilde{\gamma}'(t) = \arg f'(\gamma(t)) + \arg \gamma'(t) \in \mathbb{Z}/2\pi\mathbb{Z}.$$

At $t = t_0$, the argument of $\tilde{\gamma}(t_0)$ equals $\arg \gamma'(t)$+ a fixed number $(\arg f'(\gamma(t_0)))$.
Therefore, if γ_1, γ_2 are two curves with $\gamma_1(t_0) = \gamma_2(t_0) = z_0$, if $f'(z_0) \neq 0$, and $\gamma_1'(t_0), \gamma_2'(t_0) \neq 0$, then

$$\arg \tilde{\gamma}_1 - \arg \tilde{\gamma}_2 = \arg \gamma_1 - \arg \gamma 2. \tag{**}$$

In other words, the angle is indeed preserved by an analytic function.

Theorem 2.3.3

Suppose $f : \Omega \to \mathbb{C}$ is analytic. Then f preserves the angles at every point $z_0 \in \Omega$ where f' does not vanish.

Also, observe that for every γ and $\tilde{\gamma} = f \circ g$,

$$|\tilde{\gamma}'(f(z_0))| = |f'(z_0)||\gamma'(t_0)|,$$

so the analytic function also multiplies $|\gamma'(t_0)|$ by a fixed constant, in this case $|f'(z_0)|$.

Definition 2.3.4: Conformal Mapping

A function $f : \Omega \to \mathbb{C}$ which preserves angles in the sense (**) and such that

$$\lim_{z \to z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|}$$

exists for every $z_0 \in \Omega$ is called **conformal**.

It follows from this definition that analytic functions are conformal at all points where the derivative does not vanish.

Example 2.3.5: (Angle Preservation) $\not\Rightarrow$ (Conformality). The function $f(z) = \overline{z}$ is not conformal even though it preserves the size of the angles. Similarly, $\overline{f(z)}$, where f is analytic, preserves the **size** of the angles of every point where f' is nonzero.

Example 2.3.6. $f(z) = z^2$ doubles the angle at 0. Similarly, $z \mapsto z^m$, where $m \in \mathbb{N}$, multiplies the angle by m at origin.

We will show later that if $f'(z) = 0$ and $f \not\equiv$ constant then f multiplies the angle between curves by an integer given by the multiplicity of the zero of $f(z) - f(z_0)$.

Now, for the converse (conformal) $\Rightarrow$ (analytic), assume that $f : \Omega \to \mathbb{C}$ (not assumed to be analytic) is C^1, i.e., $\partial f/\partial x, \partial f/\partial y$ are continuous, and f preserves the angles (argument) between the curves. Let

$$\tilde{\gamma}(t) = f(\gamma(t))$$

and $\gamma(t) = \gamma_1(t) + i\gamma_2(t)$ be the decomposition of γ. Then

$$\tilde{\gamma}'(t) = f_x\gamma_1'(t) + f_y\gamma_2'(t) = f_x \cdot \frac{\gamma'(t) + \overline{\gamma}'(t)}{2} + f_y\frac{\gamma'(t) - \overline{\gamma}'(t)}{2i}$$

Let $z = \gamma'(t_0)$. Then

$$\tilde{\gamma}'(t) = f_x\frac{z + \overline{z}}{2} + f_y\frac{z - \overline{z}}{2i},$$

so

$$\tilde{\gamma}'(t) = \frac{1}{2}(f_x - if_y)z + \frac{1}{2}(f_x + if_y)\overline{z}. \tag{*}$$

Assume that $z(t_0) \neq 0$. Since the angles are assumed to be preserved and

$$\tilde{\gamma}'(t) = f'(\gamma(t))\gamma'(t),$$

we have

$$\arg(\tilde{\gamma}'/\gamma')$$

is independent of $\arg\gamma'(t_0)$. (*) implies that

$$\frac{1}{2}(f_x - if_y) + \frac{1}{2}(f_x + if_y)\frac{\overline{z}}{z} \tag{**}$$

has a constant argument. In other words we need (**) to have a constant argument regardless of $z \neq 0$. As $z \in \mathbb{C}\backslash\{0\}$, $\overline{z}/z$ is arbitrary with modulus 1, (**) will be a circle with center $(f_x - if_y)/2$ and radius $(f_x + if_y)/2$. Therefore the modulus cannot be constant unless the radius is zero, i.e., if $f_x + if_y = 0$, which is exactly what (CR) says:

$$f_x = -if_y \Leftrightarrow u_x + iv_x = -iu_y + v_y \Leftrightarrow \text{(CR)}.$$

Beginning of Jan.24, 2022

2.4 Linear Fractional Transformation

We work on $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$.

Definition 2.4.1: Linear Fractional Transformation

A mapping

$$Sz := \frac{az+b}{cz+d} \qquad a,b,c,d \in \mathbb{C}, cd \neq 0$$

is called a **linear fractional transformation**. If in addition $ad - bc \neq 0$ then we call it a **Möbius transformation**. (If $ad = bc$ it is just a constant mapping assuming it is well-defined.)
If S is Möbius (i.e., nonconstant), then S^{-1} is also a fractional transformation. *See below*.

We can represent $z \mapsto Sz = (az+b)/(cz+d)$ via the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. To see this, if

$$S_1 z = \frac{a_1 z + b_1}{c_1 z + d_1} \qquad \text{and} \qquad S_2 z = \frac{a_2 z + b_2}{c_2 z + d_2},$$

then (just like matrix multiplication)

$$\begin{aligned} S_1 S_2 z &= \frac{a_1(a_2 z + b_2)/(c_2 z + d_2) + b_1}{c_1(a_2 z + b_2)/(c_2 z + d_2) + b_1} \\ &= \frac{(a_1 a_2 + c_2 b_1)z + (a_1 b_2 + d_2 b_1)}{(c_1 a_2 + c_2 d_1)z + (c_1 b_2 + d_2 d_1)} \\ &\sim \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} z. \end{aligned}$$

(We are not really saying that this is equivalent to matrix multiplication; we simply said that the composition resembles a pattern observed in matrix multiplications.)
From this, we also see that if S is Möbius, then S^{-1} corresponds to the inverse of the matrix as well!

Some Special Linear Transformations

(1) **Translation** is given by $\begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}$ is equivalent to $z \mapsto z + \alpha$.

(2) **Dilation** is given by $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$ with positive real number k and **rotation** is the same matrix with complex k. In both cases the transformation is $z \mapsto kz$. (In the complex case it is also $e^{ik}z$.)

(3) **Inversion** corresponds to $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $z \mapsto 1/z$.

If $c \neq 0$, we can write

$$\frac{az+b}{cz+d} = \frac{az+(c/a)d}{cz+d} + \frac{b-(c/a)d}{cz+d} = \frac{a}{c} = \frac{b-(c/da)d}{cz+d} = \frac{a}{c} = \frac{(b/c)-(1/a)d}{z+(d/c)}.$$

Then this is a composition given by translation, inversion, notation, dilation, and finally translation.
If $c = 0$ then the mapping becomes

$$z \mapsto \frac{az+b}{d} = \frac{a}{d}z + \frac{b}{d},$$

a composition of rotation, dilation, and translation.

2.5 Cross Ratio

Note that the linear transformation above has four parameters, but it only has degree of freedom 3, as we can normalize the entire equation by setting and fixing any variable to 1. We say the linear transformation has **complex degrees** of 3. Conversely —

Proposition 2.5.1

Given distinct $z_2, z_3, z_4 \in \mathbb{C}_\infty$, there exists a unique linear transformation S such that

$$Sz_2 = 1, \qquad Sz_3 = 0, \qquad \text{and } Sz_4 = \infty.$$

Proof of existence. If z_2, z_3, z_4 are all finite, then the linear transformation is simply given by

$$Sz := \left(\frac{z-z_3}{z-z_4}\right)\left(\frac{z_2-z_3}{z_2-z_4}\right)^{-1}.$$

If $z_2 = \infty$, in the above equation, intuitively $\frac{z-z_3}{z-z_4} \to 1$ as $z \to \infty$, so we simply define $Sz := \frac{z-z_3}{z-z_4}$.
If $z_3 = \infty$, define $Sz := \frac{z_2-z_4}{z-z_4}$. If $z_4 = \infty$, define $Sz := \frac{z-z_3}{z_2-z_3}$. □

We define the **cross ratio** to be

$$(z, z_2, z_3, z_4) := \frac{z-z_3}{z-z_4} \bigg/ \frac{z_2-z_3}{z_2-z_4}.$$

In fact, we can map different z_2, z_3, z_4 to arbitrary distinct w_2, w_3, w_4 by composing with the *inverse* of

$$Tz = \frac{(w-w_3)(w_2-w_4)}{(w-w_4)(w_2-w_3)}.$$

(The inverse maps $1 \mapsto w_2, 0 \mapsto w_3$, and $\infty \mapsto w_4$.)

Proof of uniqueness. Let T be another mapping also satisfy the conditions. Then $ST^{-1}(1) = 1$, $ST^{-1}(0) = 0$, and $ST^{-1}(\infty) = \infty$. Let $ST^{-1}z := \dfrac{az+b}{cz+d}$ (this is well-defined because we know S^{-1} is fractional and ST^{-1} is therefore fractional). The three conditions imply

$$a + b = c + d \qquad b = 0 \qquad \text{and } c = 0,$$

so $a = d$ and $b = c$, i.e., $T = S$. □

Definition 2.5.2: Cross Ratio

The **cross ratio** (z_1, z_2, z_3, z_4), where $\{z_2, z_3, z_4\}$ are pairwise distinct, is the map of z, under S a linear fractional transformation, defined by

$$z_2 \mapsto 1, \qquad z_3 \mapsto 0, \qquad \text{and } z_4 \mapsto \infty.$$

From the previous remark, such transformation is given by

$$(z_1, z_2, z_3, z_4) := \frac{z_1 - z_3}{z_1 - z_4} \Big/ \frac{z_2 - z_3}{z_2 - z_4}.$$

(If one of them is infinite, adjust correspondingly as we did before.)

Beginning of Jan.26, 2022

Proposition 2.5.3

If z_1, z_2, z_3, z_4 are distinct and if T is Möbius, then

$$(Tz_1, Tz_2, Tz_3, Tz_4) = (z_1, z_2, z_3, z_4),$$

i.e., linear mappings preserve cross ratios.

Proof. Let $Sz = (z_1, z_2, z_3, z_4)$, i.e., the mapping satisfying $z_2 \mapsto 1, z_3 \mapsto 0, z_4 \mapsto \infty$. Then

$$ST^{-1} : Tz_2 \mapsto 1, Tz_3 \mapsto 0, Tz_4 \mapsto \infty.$$

Therefore, $(Tz_1, Tz_2, Tz_3, Tz_4) = (ST^{-1})(Tz_1) = Sz_1 = (z_1, z_2, z_3, z_4)$. □

Proposition 2.5.4

Let z_1, z_2, z_3 be distinct and let w_1, w_2, w_3 be distinct as well. Then we can map $z_1 \mapsto w_1, z_2 \mapsto w_2, z_3 \mapsto w_3$ via $(w, w_1, w_2, w_3) = (z, z_1, z_2, z_3)$.

Proof. Let T be Möbius so that $(T_z, Tz_1, Tz_2, Tz_3) = (z, z_1, z_2, z_3)$. Then $(Tz, w_1, w_2, w_3) = (z, z_1, z_2, z_3)$ and we are done. □

Theorem 2.5.5

Let z_1, z_2, z_3, z_4 be distinct. Then $(z_1, z_2, z_3, z_4) \in \mathbb{R}$ if and only if z_1, z_2, z_3, z_4 lie on a circle or a line.

When studying linear (fractional) transformations, we consider this as circle going through ∞.

Proof. It is sufficient to prove that a linear transformation maps $\mathbb{R}$ to a circle on a line. Let $Tz = (az+b)/(cd+z)$. Since $Tz \in \mathbb{R}$, we have

$$\frac{az+b}{cz+d} = \frac{\overline{a}\overline{z}+\overline{b}}{\overline{c}\overline{z}+\overline{d}} \implies (a\overline{c}-\overline{a}c)z\overline{z} + (a\overline{d}-\overline{b}c)z + (b\overline{c}-\overline{a}d)\overline{z} + b\overline{d} - \overline{b}d = 0.$$

If $a\overline{c}-\overline{a}c = 0$ then we get something like $(a\overline{d}-\overline{b}c)z + (b\overline{c}-\overline{a}d)\overline{z} + b\overline{d} - \overline{b}d = 0$. This either an empty set, a point, or a line. Since it must be an infinite set (it is the image of $\mathbb{R}$ under the inverse) we exclude the possibility of a point. If $a\overline{c}-\overline{a}c \neq 0$, we show that z_1, z_2, z_3, z_4 lie on a circle. We divide everything by $a\overline{c}-\overline{a}c$ and get

$$|z|^2 + \frac{a\overline{d}-c\overline{b}}{a\overline{c}-\overline{a}c}z + \frac{b\overline{c}-\overline{a}d}{a\overline{c}-\overline{a}c}\overline{z} + \frac{b\overline{d}-\overline{b}d}{a\overline{c}-\overline{a}c} = 0.$$

Completing the square, we have

$$\left(z + \frac{\overline{a}d-\overline{c}b}{\overline{a}c-a\overline{c}}\right)\left(\overline{z} + \frac{a\overline{d}-c\overline{b}}{a\overline{c}-\overline{a}c}\right) = \frac{\overline{b}d-b\overline{d}}{a\overline{c}-\overline{a}c} - \frac{(\overline{a}d-\overline{c}b)(a\overline{d}-c\overline{b})}{(\overline{a}c-a\overline{c})^2}.$$

The RHS can be expanded:

$$\text{RHS} = \frac{(\overline{b}d-b\overline{d})(a\overline{c}-\overline{a}c) - (\overline{a}d-\overline{c}b)(a\overline{d}-c\overline{b})}{(a\overline{c}-\overline{a}c)(\overline{a}c-a\overline{c})^2} = \ldots = \left|\frac{ad-bc}{\overline{a}c-\overline{c}a}\right|^2.$$

Therefore,

$$\left|z + \frac{\overline{a}d-\overline{c}b}{\overline{a}c-a\overline{c}}\right|^2 = \left|\frac{ad-bc}{\overline{a}c-\overline{c}a}\right|.$$

Taking square roots we see z must be some constant distance from some point, i.e., z must lie on a circle. □

Corollary 2.5.6

A Möbius transformation maps circles to circles. (We showed that real axis can go to circles, so applying the inverse once again gives our desired circle-to-circle mapping.)

Definition 2.5.7

z and z^* are **symmetric** with respect to the circle through z_1, z_2, z_3 if

$$(z^*, z_1, z_2, z_3) = \overline{(z, z_1, z_2, z_3)}.$$

Remark. Right now the definition depends on the choice of z_1, z_2, z_3, but in fact it is independent, as we will prove later.

Beginning of Jan.28, 2022

> **Remark.** Note that the operation defined in symmetry is also symmetric:
> $$(z^*, z_1, z_2, z_3) = \overline{(z, z_1, z_2, z_3)} \iff \overline{(z^*, z_1, z_2, z_3)} = (z, z_1, z_2, z_3).$$

Since linear transformations preserve the cross-ratio, we can reduce the case to where $C = \mathbb{R}$, since we can map a circle to the real axis.

Then $z_1, z_2, z_3 \in \mathbb{R}$, and the symmetry means

$$\frac{\frac{z^*-z_2}{z^*-z_3}}{\frac{z_1-z_2}{z_1-z_3}} = \frac{\frac{\overline{z}-z_2}{\overline{z}-z_3}}{\frac{z_1-z_2}{z_1-z_3}} \implies \frac{z^*-z_2}{z^*-z_3} = \frac{\overline{z}-z_2}{\overline{z}-z_3}.$$

Since Möboius transformations (check it is Möbius) are injective, we must have $z^* = \overline{z}$.

This shows that z^* is *uniquely* determined by z and it does *not* depend on the choice of z_1, z_2, z_3.

Also, recall that dilation and rotation are Möbius. Therefore if z_1, z_2, z_3 lie on any other line, z^* is the mirror image across that line.

How about symmetry with respect to circles?

WLOG assume the circle is centered at origin and has radius 1, denoted $\mathbb{D} := D_1(0) = B_1(0)$. By rotation, we further assume $z \in (0, 1)$. We define a mapping $T : \mathbb{H} \to \mathbb{D}$ (where $\mathbb{H}$ denotes the upper half plane, i.e., z with $\mathfrak{Im} z > 0$) by

$$z \mapsto \frac{z-i}{z+i} \qquad \text{and } z \mapsto i\frac{1+z}{1-z}.$$

This is a **conformal** (bijective and analytic) mapping from $\mathbb{H}$ to $\mathbb{D}$. For convenience we call the inverse S.

Then

$$a^* = S^{-1}\left(-i\left(\frac{1+a}{1-a}\right)\right) = \frac{-i(1+a)/(1-a)-i}{-i(1+a)/(1-a)+i} = \frac{1}{a},$$

i.e., the symmetric point for a is $1/a$. The same claim holds after rotation and dilation, except we rotate the way from $\mathbb{R}^+$ to the one that originates at the center and passes through z. Note that we have $|z^*||z| = r^2$. As $z \to$ the center, $z^* \to$ infinity, and as $|z| \to r$ we have $|z| \to r$ as well.

Two Important Conformal Mappings

The first one is $T : \mathbb{H} \to \mathbb{D}$ as defined above: $z \mapsto \dfrac{z-i}{z+i}$.

The other one: $z \mapsto \dfrac{z-1}{z+1}$, which maps $\{\mathfrak{Re} z > 0\} \to \mathbb{D}$.

To understand the Möbius mappings, consider

$$Tz = k\frac{z-a}{z-b}.$$

Then the line segment connecting a to b becomes a ray starting from 0. The circle arcs connecting a to b become rays oriented in other directions (since $b \mapsto \infty$ and circles go to circles, they must be mapped to lines!). In other words, arcs go to rays.

On the other hand, what's the preimage of circles? If $|k|\left|\dfrac{z-a}{z-b}\right| = r$ then

$$\frac{|z-a|}{|z-b|} = \frac{r}{|k|},$$

i.e., the ratio of distance between $z \to a$ and $z \to b$ are kept constant. These are circles centered somewhere on the line passing through a and b. These are called **Apollonius circles**.

Chapter 3

Complex Integration

Beginning of Jan.31, 2022

Assume that $\gamma : [a, b] \to \Omega$ (where $\Omega \subset \mathbb{C}$ is open) is piecewise smooth. Then, for $f : \Omega \to \mathbb{C}$, we define

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t$$

We immediately have some nice properties:

(1) Invariance under change of parameter: if $t = t(\tau)$ where $\tau : [\alpha, \beta] \to [a, b]$ is piecewise smooth and γ smooth, then

$$\int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t = \int_\alpha^\beta f(\gamma(t(\tau)))\gamma'(t(\tau))t'(\tau)\,\mathrm{d}\tau.$$

(2) We can have directed integrals, which can be proved via reparametrization:

$$\int_{-\gamma} f(z)\,\mathrm{d}z = -\int_\gamma f(z)\,\mathrm{d}z.$$

(3) Partition: if γ can be partitioned into $\gamma_1, \gamma_2, \ldots, \gamma_n$, then

$$\int_\gamma f(z)\,\mathrm{d}z = \sum_{i=1}^n \int_{\gamma_i} f(z)\,\mathrm{d}z.$$

(4) Integral w.r.t. conjugate:

$$\int_\gamma f(z)\,\mathrm{d}z = \overline{\int_\gamma \overline{f(z)}\,\mathrm{d}\bar{z}}.$$

(5) Decomposition:

$$\int_\gamma f(z)\,\mathrm{d}x = \int_\gamma \mathfrak{Re} f(x)\,\mathrm{d}x + i\int_\gamma \mathfrak{Im} f(x)\,\mathrm{d}x.$$

Similar things for $\mathrm{d}y$. Alternatively,

$$\int_\gamma f\,\mathrm{d}x = \frac{1}{2}\int_\gamma f\,\mathrm{d}z + \frac{1}{2}\int_\gamma f\,\overline{\mathrm{d}z}$$

and

$$\int_\gamma f\,\mathrm{d}y = \frac{1}{2i}\int_\gamma f\,\mathrm{d}z - \frac{1}{2i}\int_\gamma f\,\overline{\mathrm{d}z}.$$

Definition 3.0.1: Arc Length

Let $\gamma : [a, b] \to \Omega$ be piecewise smooth. Then we define

$$\int_\gamma f\,|\mathrm{d}z| := \int_\gamma f(z(t))|z'(t)|\,\mathrm{d}t$$

is called the **arc length** integral.

Theorem 3.0.2

Let p, q be continuous on Ω where $\Omega \subset C$ is open and connected. Then

$$\int_\gamma (p\,\mathrm{d}x + q\,\mathrm{d}y)$$

depends only on the endpoints of γ if and only if there exists $U \in C^1(\Omega)$ such that

$$\frac{\partial U}{\partial x} = p \qquad \text{and} \qquad \frac{\partial U}{\partial y} = q.$$

Proof. (The proof is identical to its calculus counterpart.) Assume that there exists such U and let γ be parametrized by some $(x(t), y(t))$, $t \in [a, b]$. By definition,

$$\begin{aligned}\int_\gamma (p\,\mathrm{d}x + q\,\mathrm{d}y) &= \int_\gamma \left(\frac{\partial U}{\partial x}\,\mathrm{d}x + \frac{\partial U}{\partial y}\,\mathrm{d}y\right)\\ &= \int_a^b \left(\frac{\partial U}{\partial x}x'(t) + \frac{\partial U}{\partial y}y'(t)\right)\,\mathrm{d}t\\ &= \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}\left(U(x(t), y(t))\right)\,\mathrm{d}t\\ &= U(x(b), y(b)) - U(x(a), y(a)).\end{aligned}$$

Conversely, we assume that the integral depends only on the endpoints. Fix (x_0, y_0) and define

$$U(x, y) := \int_\gamma (p\,\mathrm{d}x + q\,\mathrm{d}y)$$

where γ is any curve starting at (x_0, y_0) and ends at (x, y).Consider the *polygonal curve* with segments parallel to the x, y axis and ending with the horizontal part. (This is possible since U is open and connected.) Denote the last segment by (x_1, y) and (x, y). Then we have

$$U(x, y) = U(x_1, y) + \int_{x_1}^x p(s, y)\,\mathrm{d}s.$$

Then,

$$\frac{\partial U}{\partial x} = p(x, y).$$

Similarly, we can use a polygonal curve with an vertical ending piece to show that $\dfrac{\partial U}{\partial y} = q(x, y)$. □

Definition 3.0.3

Let $\Omega \subset \mathbb{C}$ be open. We call

$$p \, \mathrm{d}x + q \, \mathrm{d}y$$

an **exact differential** if there exists $U \in C^1(\Omega)$ such that $\partial U / \partial x = p$ and $\partial U / \partial y = q$.

Now let $f(z)$ be continuous and complex valued. Assume that $f(z) \, \mathrm{d}z$ is an exact differential, where

$$f(z)\mathrm{d}z = f(z)\mathrm{d}x + if(z)\mathrm{d}y.$$

Then, there exists $F \in C^1(\Omega, \mathbb{C})$ such that $\partial F / \partial x = f(z)$ and $\partial F / \partial y = if(z)$. Observe that

$$\frac{\partial F}{\partial x} = -i \frac{\partial F}{\partial y},$$

which is the **Cauchy-Riemann equation in complex form**.
Check: if $F = U + iV$ then $F_x = U_x + iV_x$ and $F_y = U_y + iV_y$. Tthus

$$\frac{\partial F}{\partial x} = -i \frac{\partial F}{\partial y} \iff U_x + iV_x = -iV_y + V_y \iff U_x = V_y \text{ and } V_x = -U_y..$$

That is, if $f\mathrm{d}z$ is an exact differential, then it is the derivative of an analytic function.

Beginning of Feb.2, 2022

Theorem 3.0.4

Let f be a complex-valued continuous function on Ω. Then $\int_\gamma f(z) \, \mathrm{d}z$ depends only on endpoints of γ if and only if there exists F analytic on Ω such that $F' = f$.
This criterion is equivalent to $\int_\gamma f(z) \, \mathrm{d}z = 0$ for every closed, piecewise smooth γ.
(At this point we don't conclude if f is analytic – we will later show that it is though.)

Corollary 3.0.5

$\int_\gamma (z-a)^n \, \mathrm{d}z = 0$ for all $a \in \mathbb{C}$ and for any closed piecewise smooth γ in $\mathbb{C}$ since $(z-a)^n$ is the derivative of $(z-a)^{n+1}/(n+1)$.

Proposition 3.0.6

Let C be a (positive oriented) circle C around $a \in \mathbb{C}$ with radius $\rho > 0$. Then

$$\int_C \frac{1}{z-a} \, \mathrm{d}z = 2\pi i$$

and

$$\int_C \frac{1}{(z-a)^n} \, \mathrm{d}z = 0 \qquad \text{for } n \geqslant 2.$$

Proof. We use $z(t) = a + \rho e^{iz}$, $0 \leqslant \theta \leqslant 2\pi$ as the parametrization for C. Then by definition

$$\int_C \frac{1}{z-a}\,\mathrm{d}z = \int_0^{2\pi} \frac{\rho i e^{i\theta}}{\rho e^{i\theta}}\,\mathrm{d}\theta - \int_0^{\pi} i\,\mathrm{d}\theta = 2\pi i.$$

□

Theorem 3.0.7: Cauchy's Theorem for Rectangles

Let $R = [a,b] \times [c,d]$ be a rectangle with boundary ∂R given by $(a,c) \to (b,c) \to (b,d) \to (a,d)$ where $a,b,c,d < \infty$, $a < b$, and $c < d$. If f is analytic in (a neighborhood of) R, then

$$\int_{\partial R} f(z)\,\mathrm{d}z = 0.$$

Proof. For any rectangle S, denote $\eta(S) := \displaystyle\int_{\partial S} f(z)\,\mathrm{d}z$. We divide R into 4 congruent rectangles by two perpendicular bisectors. Among these four, choose the rectangle with the biggest η. Call this R_1 (so in particular $|\eta(R_1)| \geqslant |\eta(R)|/4$). We further divide R_1 into four congruent rectangles and choose R_2 with $|\eta(R_2)| \geqslant |\eta(R_2)|/4$. Continuing by induction, we obtain a sequence $\{R_n\}$ with $|\eta(R_{n+1})| \geqslant |\eta(R_n)|/4$. Also, the perimeters satisfy $|\partial R_{n+1}| = |\partial R_n|/2$. Furthermore, there exists $z^* \in \mathbb{C}$ with $\displaystyle\bigcap_{n=1}^{\infty} R_n = \{z^*\}$.
Let $\epsilon > 0$. Then b assumption there exists $\delta > 0$ such that

$$|z - z^*| < \delta \implies \left|\frac{f(z) - f(z^*)}{z - z^*} - f'(z^*)\right| < \epsilon.$$

That is,

$$|f(z) - f(z^*) - (z - z^*)f'(z^*)| \leqslant \epsilon|z - z^*|.$$

We choose $n_0 \in \mathbb{N}$ such that $R_n \subset B(z^*, \delta)$ for all $n \geqslant n_0$. Then

$$\begin{aligned}\eta(R_n) &= \int_{\partial R_n} f(z)\,\mathrm{d}z \\ &= \int_{\partial R_n} f(z) - f(z^*) - (z - z^*)f'(z^*)\,\mathrm{d}z\end{aligned}$$

(since integrating a constant $f(z^*)$ over a closed curve is zero.) Thus

$$|\eta(R_n)| \leqslant \int_{\partial R_n} \epsilon|z - z^*|\,|\mathrm{d}z| \leqslant \epsilon|\partial R_n||\partial R_n|.$$

Then, $|\eta(R)| \leqslant 4^n|\eta(R_n)| \leqslant 4^n\epsilon|\partial R_n|^2 = \epsilon|\eta(R)|^2$. Therefore $|\eta(R)|$ must be zero! □

Theorem 3.0.8

Let R be a rectangle and let $z_1, \ldots, z_n$ be distinct in the interior of R. Assume f is analytic in $R' := R\backslash\{z_1, \ldots, z_n\}$ and assume that $\lim_{z \to z_i}(z - z_i)f(z) = 0$.
Then

$$\int_{\partial R} f(z)\,\mathrm{d}z = 0.$$

In particular, we can define/change of values of f at these points to make f analytic on R.

Beginning of Feb.4, 2022

In fact, a sufficient condition for $\lim_{z\to z_i}(z-z_i)f(z)=0$ is that f is bounded in R'.

Proof. WLOG assume $n=1$ (for otherwise we can subdivide the rectangles). Let $\epsilon>0$. Then there exists $\delta>0$ such that

$$|z-z_1|<\delta \implies |f(z)|\leqslant \frac{\epsilon}{|z-z_1|}.$$

Then we find a square R_0 centered at z_1 contained in $B_\delta(z_1)$. Then

$$\left|\int_{\partial R_0} f\,\mathrm{d}z\right|\leqslant \epsilon\int_{\partial R_0}\frac{|\mathrm{d}z|}{|z-z_1|}\leqslant \frac{\epsilon}{\min\limits_{z\in\partial R_0}|z-z_1|}|\partial R_0|\leqslant C\epsilon.$$

Now, subdividing R into nine rectangles, with the middle one being R_0, we see

$$\int_{\partial R_0} f\,\mathrm{d}z=\int_{\partial R} f\,\mathrm{d}z.$$

Since ϵ is arbitrary (even though R_0 depends on it), we are done. □

Theorem 3.0.9

Assume that $f(z)$ is analytic in $\mathbb{D}$ and γ is a closed piecewise smooth curve in $\mathbb{D}$. Then

$$\int_\gamma f(z)\,\mathrm{d}z=0.$$

Later, we will generalize this from $\mathbb{D}$ to any simply connected domain.

Proof. We will use a previous theorem saying that if $f=F'$ for some analytic F then the claim holds.
We define

$$F(z):=\int_{\sigma_1} f(z)\,\mathrm{d}z$$

where σ_1 is the piecewise linear curve from $0\to \mathfrak{Re}\,z\to z$ (i.e., first horizontal then vertical). Then using $F(z)=\int_{\sigma_1} f\,\mathrm{d}x+i\int_{\sigma_1} f\,\mathrm{d}y$ we have

$$\frac{\partial F}{\partial y}=if.$$

Now consider the path σ_2 given by $0\to\mathfrak{Im}\,z\to z$ (i.e., first vertical then horizontal). Then

$$F(z)=\int_{\sigma_2} f\,\mathrm{d}z$$

since $\sigma_1\to(-\sigma_2)$ forms a rectangle and by the Cauchy's theorem, $\int_{-\sigma_2} f\,\mathrm{d}z=-F(z)$. Therefore,

$$\frac{\partial F}{\partial x}=f.$$

Therefore

$$\frac{\partial F}{\partial x}=-i\frac{\partial F}{\partial y}$$

and (C-R) in complex form implies F is analytic. □

Theorem 3.0.10

Let $z_1, z_2, ..., z_n \subset \mathbb{D}$, assume f is analytic on $\Omega := \mathbb{D} \backslash \{z_1, ..., z_n\}$, with

$$\lim_{z \to z_j} (z - z_j) f(z) = 0 \qquad \text{for } j = 1, ..., n.$$

Then

$$\int_\gamma f(z) \, \mathrm{d}z = 0 \qquad \text{for all closed, piecewise smooth } \gamma \subset \Omega.$$

Proof. We pick $z_0 \in \Omega$ such that the vertical and horizontal lines through z_0 contain no exceptional point $(z_1, ..., z_n)$. Then we use two paths along with the "rectangle-with-dot" theorem to prove this claim.
For $z \in \Omega$, we pick $\tilde{z}$ close to z and let σ_1, σ_2 be the corresponding "zigzag" paths from z_0 to $\tilde{z}$ and then to z, while both σ_1, σ_2 begin by $z_0 \to z_0 + \mathfrak{Re}(\tilde{z} - z_0) \to \tilde{z}$ such that this sub-path contains no singularity. Again define

$$F(z) := \int_{\sigma_1} f \, \mathrm{d}z = \int_{\sigma_2} f \, \mathrm{d}z.$$

By a same reasoning we will see that F is analytic which concludes the proof. □

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Proof. Let $A : \{z \in \mathbb{D} : \mathfrak{Im} z \neq \mathfrak{Im} z_i, \mathfrak{Re} z \neq \mathfrak{Re} z_i\}$, i.e., the collection of "good" points whose corresponding vertical and horizontal lines contain no singular points. We choose $z_0 \in A$. For $z \in \Omega$, choose any vertical-horizontal-vertical path σ_1 avoiding $z_1, ..., z_n$.
Note that the definition $F(z) := \int_{\sigma_1} f(z) \, \mathrm{d}z$ is well-defined and independent of choice of the horizontal part of σ_1. This is because of the "generalized" Cauchy's rectangle theorem which states that the integral around a rectangle, even if there are bad points inside, is 0.
Because of this, the y-derivative of F depends only on the last vertical segment, that is,

$$\frac{\partial F}{\partial y} = if.$$

Similarly, we define σ_2 to be a horizontal-vertical-horizontal path, also avoiding $z_1, ..., z_n$, and by the same token

$$\frac{\partial F}{\partial x} = f.$$

(We can easily check that with the same endpoinds, integral over σ_1 and σ_2 are indeed the same by using the rectangle theorem twice, so the resulting capital function is indeed F). Then we have

$$\frac{\partial F}{\partial x} = -i \frac{\partial F}{\partial y}$$

so F is analytic. It remains to notice that

$$F' = \frac{\partial \mathfrak{Re} F}{\partial x} + i \frac{\partial \mathfrak{Im} F}{\partial x} = \frac{\partial \mathfrak{Re} F}{\partial x} - i \frac{\partial \mathfrak{Re} F}{\partial y} = \mathfrak{Re} f - i(-\mathfrak{Im} f) = \mathfrak{Re} f + i \mathfrak{Im} f = f.$$

□

Upshot. If F is analytic and $\dfrac{\partial F}{\partial x} = f, \dfrac{\partial F}{\partial y} = if$, then $F'(z) = f(z)$.

3.1 Index / Winding Number

To define the index of a curve around a point, we need the next lemma:

Lemma 3.1.1

Let $a \in \mathbb{C}$ and let γ be a piecewise smooth *closed* curve that does *not* pass through a. Then

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{z-a}\,\mathrm{d}z \in \mathbb{Z}.$$

Recall that γ is oriented; that is, reversing the orientation flips the sign of the number above and therefore the index. We assume γ is piecewise smooth.

Definition 3.1.2: Index

For $a \in \mathbb{C}$ not on the curve $\{\gamma\}$ (the image of $\gamma : [\alpha, \beta] \to \mathbb{C}$), we define the **index / winding number** of γ with respect to a as

$$n(\gamma, a) := \frac{1}{2\pi i}\int_\gamma \frac{1}{z-a}\,\mathrm{d}z.$$

Motivation for winding number.

If $\log$ exists on some γ (not necessarily closed), then

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{z-a}\,\mathrm{d}z = \frac{1}{2\pi i}(\log(z_2-a)-\log(z_1-a)) = \frac{1}{2\pi i}\log\frac{|z_2-a|}{|z_1-a|} + \frac{1}{2\pi}(\arg z_2 - \arg z_1).$$

As γ approaches a closed curve, $\arg z_2 - \arg z_1$ approaches $\pm 2\pi$, and the first quotient $\to 0$. That is,

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Proof of Lemma. Let $z(t)$, $t \in [\alpha, \beta]$, be a parametrization of γ. Let

$$h(t) := \int_\alpha^t \frac{z'(s)}{z(t)-a}\,\mathrm{d}s,$$

which is well-defined since $z(t) \neq a$ for all t.

Idea: we expect $h(t) = \log(z(t)-a)$ but we have some technical difficulties. Thus we consider the exponential. Then the fundamental theorem of calculus should imply that $e^{-h(t)}(z(t)-a)$ is the constant 1 so derivative of everything is 0.

We have, by FTC, that

$$h'(t) = \frac{z'(t)}{z(t)-a}$$

(for piecewise smooth curve, partition the interval if needed). Then

$$\begin{aligned}(e^{-h(t)}(z(t)-a))' &= -h'(t)e^{-h(t)}(z(t)-a) + e^{-h(t)}z'(t)\\ &= -\frac{z'(t)}{z(t)-a}e^{-h(t)}(z(t)-a) + e^{-h(t)}z'(t) = 0.\end{aligned}$$

This implies $e^{-h(t)}(z(t)-a)$ is constant on each partitioned interval. But $e^{-h(t)}(z(t)-a)$ itself is continuous, so it must be constant everywhere on $[\alpha, \beta]$. Hence substituting t by α gives

$$e^{-h(t)}(z(t)-\alpha) = e^{-h(t)}(z(\alpha)-a) = e^{-h(\alpha)}(z(\alpha)-a) = e^{-h(\beta)}(z(\beta)-a)$$

so

$$e^{-h(t)} = e^{h(\alpha)} \frac{z(\alpha) - a}{z(t) - a} \qquad \text{and} \qquad e^{-h(\beta)} = \frac{z(\alpha) - a}{z(\beta) - a}.$$

(Note that $h(\alpha) = 0$.) Since $z(\alpha) = z(\beta)$ (the curve is closed) we see $e^{-h(\beta)} = 1$, so $-h(\beta) \in 2\pi i\mathbb{Z}$. □

Proposition 3.1.3

Assume that γ is inside $B_r(a)$. Then $n(\gamma, z_0) = 0$ for all $z_0 \notin B_r(a)$.

Proof. Since $1/(z - z_0)$ is analytic in $B_r(a)$. Then this along with γ being closed implies the integral being 0. □

If γ is closed, then $\mathbb{C} \backslash \{\gamma\}$ is open, so it is the union of open connected sets. These are the **regions** determined by γ.

Proposition 3.1.4

The index $n(\gamma, a)$ is constant in each of the regions determined by γ and is 0 in the unbounded region.

Proof. The index is a continuous function in z_0 for z in the region determined by γ (a short $\epsilon - \delta$ proof on the definition suffices) but the index can only take integer values. For the unbounded region, use the previous result — continuously transform a to a sufficiently far a' so that γ can be entirely contained in some disk not including a'. □

Lemma 3.1.5

Let γ be closed. Let $z_1, z_2 \in \{\gamma\}$ and assume $0 \notin \{\gamma\}$. Let γ_1 be the part of γ going from z_1 to z_2 and let γ_2 be the part from z_2 to z_1. Suppose

$$\mathfrak{Im} z_2 > 0 > \mathfrak{Im} z_1$$

and assume

$$\gamma_1 \cap (-\infty, 0) = r_2 \cap (0, \infty) = \varnothing.$$

Then $n(\gamma, 0) = 1$.

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Proof. Let C be a "small" circle around 0 not touching γ (in fact this is redundant). Consider the rays originating from origin and passing though z_1 and z_2. They intersect C at two points. Let the one corresponding to $z_1 \to z_2$ be C_1 and the other by C_2. Let the ray originating from z_1 to the starting of C_1 be δ_1 and the other δ_2. Consider

$$\sigma_1 := \gamma_1 + \gamma_2 - c_1 - \delta_1 \qquad \text{and} \qquad \sigma_2 = \gamma_2 + \sigma_2 - c_2 - \delta_2.$$

Then

$$\begin{aligned}\gamma &= \gamma_1 + \gamma_2 \\ &= (\sigma_1 - \delta_2 + c_1 + \delta_1) + (\sigma_2 - \delta_1 + c_2 + \delta_2).\end{aligned}$$

Though γ_1, γ_2 are not closed, we can define their "non-integer index" as the integral

$$n(\gamma_j, 0) = \frac{1}{2\pi i} \int_{\gamma_j} \frac{1}{z} \, \mathrm{d}z.$$

Using cancellations,

$$n(\gamma,0) = n(C,0) + n(\sigma_1,0) + n(\sigma_2,0)$$

Note that the origin is in the unbounded component of σ_1 so $n(\sigma_1,0) = 0$. Likewise $n(\sigma_2,0) = 0$. Since $n(C,0) = 1$, we have $n(\gamma,0) = 1$. □

Alhfor's proof (HW): a piecewise piecewise smooth Jordan curve (closed path without self-intersections) splits a plane into at least two components, an unbounded one and at least one bounded one with index ±1.

3.2 Cauchy Integral Formula

Theorem 3.2.1: Local Cauchy Integral Formula

Assume f is analytic in a disk $D = D_r(a)$. Let γ be a closed curve (smooth, by convention) in D. Then

$$n(\gamma,z)f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta \qquad \text{for } z \in D\backslash\{\gamma\}.$$

In particular, if $n(\gamma,z) = 1$ for some z, then

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta.$$

We frequently apply this theorem to γ as circles.

Proof. Consider

$$F(\zeta) := \frac{f(\zeta) - f(z)}{\zeta - z} \qquad \text{for } \zeta \in D\backslash\{\gamma\}.$$

The F is analytic in $D\backslash\{z\}$. At z, we have

$$\lim_{\zeta\to z}(\zeta - z)F(\zeta) = \lim_{\zeta\to z} f(\zeta) - f(z) = 0$$

by continuity of f. Applying the "general form" of Cauchy's rectangle/disk theroem, we have

$$\int_\gamma F(\zeta)\,\mathrm{d}\zeta = \int_\gamma \frac{f(\zeta) - f(z)}{\zeta - z}\,\mathrm{d}\zeta = 0 \qquad \text{for } z \notin \{\gamma\}.$$

Therefore, for $z \notin \{\gamma\}$,

$$f(z)\int_\gamma \frac{1}{\zeta - z}\,\mathrm{d}\zeta = \int_\gamma \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta.$$

The LHS is $f(z)n(\gamma,z)$ and we are done. □

Theorem 3.2.2

Assume that f is analytic in Ω. Let $D := D_r(a)$ be such that $\overline{D} \subset \Omega$. Then f‘ is infinitely differentiable in D and for all $z \in D$,

$$f^{(n)}(z) = \frac{\mathrm{d}^n f(z)}{\mathrm{d}z^n} = \frac{n!}{2\pi i}\int_{\partial D} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\,\mathrm{d}\zeta.$$

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Proof. One can use Lebesgue differentiation theorem to show this proof, but we will adopt a more elementary one. Let $\gamma : \partial D \to \mathbb{C}$ be continuous and let

$$F_n(z) = \int_{\partial D} \frac{\gamma(\xi)}{(\xi - z)^n} \,\mathrm{d}\xi$$

where $n \in \mathbb{N}$. We claim that F_n is (continuous and) differentiable. For this, we note that for $z \in D$,

$$F(z) - F(z_0) = \int_{\partial D} \gamma(\xi) \left(\frac{1}{(\xi - z)^n} - \frac{1}{(\xi - z_0)^n} \right) \mathrm{d}\xi.$$

Since $1/(\xi - z)^n$ is uniformly continuous inside D and $z \in D$, we can adopt an $\epsilon - \delta$ argument showing that $\lim_{z \to z_0} (F(z) - F(z_0)) = 0$. (Because we also have ∂D is compact so $\sup|\varphi| < \infty$). To compute the derivative, note that $(a-b)^n = (a-b)(a^{n-1} + a^{n-2}b + \ldots + b^{n-1})$.

$$\frac{1/(\zeta - z)^n - 1/(\zeta - z_0)^n}{z - z_0} = \frac{1/(\zeta - z) - 1/(\zeta - z_0)}{z - z_0} \left(\frac{1}{(\zeta - z)^{n-1}} + \ldots + \frac{1}{(\zeta - z_0)^{n-1}} \right).$$

Note that as $z \to z_0$, every term in the second parenthesis goes to $1/(\zeta - z)^{n-1}$. In the first term we have $1/((\zeta - z)(\zeta - z_0))$. Therefore the difference quotient converges uniformly to

$$\frac{1}{(\zeta - z_0)^2} \cdot \frac{n}{(\zeta - z_0)^{n-1}}$$

Applying $\epsilon - \delta$, we have

$$\lim_{z \to z_0} \frac{F_n(z) - F_n(z_0)}{z - z_0} = \int_{\partial D} \gamma(\zeta) \frac{n}{(\zeta - z_0)^{n+1}} \,\mathrm{d}\xi.$$

□

Theorem 3.2.3: Morera's Theorem

If $f : \Omega \to \mathbb{C}$ is continuous and if $\int_\gamma f \,\mathrm{d}z = 0$ for every closed γ in Ω, then f is analytic in Ω.

Proof. We have proven that under these assumptions there exists F analytic with $F'(z) = f(z)$, i.e., f has an analytic primitive. Therefore, by the previous theorem, $f = F'$ is also analytic. □

Theorem 3.2.4: Cauchy Estimate

If $|f(\xi)| \leqslant m$ on $D_r(a)$, then $|f^{(n)}(a)| \leqslant Mn!/r^n$.

From this, f can be developed into an infinite Taylor series with radius at least r.

Proof. Choose any $0 < \rho < r$. Then

$$\frac{f^{(n)}(z)}{n!} = \frac{1}{2\pi i} \int_{\partial D_\rho(a)} \frac{f(\zeta)}{(\zeta - z)^{n+1}} \,\mathrm{d}\zeta.$$

Thus

$$\frac{|f^{(n)}(a)|}{n!} \leqslant \frac{1}{2\pi} 2\pi\rho \frac{M}{\rho^{n+1}} = \frac{M}{\rho^n}.$$

For other points, we consider smaller disks centered at z so that they are still contained in $D_r(a)$. In particular we set the radius to be $r - |z|$. Then

$$|f^{(n)}(z)| \leqslant \frac{Mn!}{(r-|z|)^n}.$$

(We will later see that $f^{(n)}(z)/n! \cdot (z-a)^n$ are the Taylor coefficients.) □

Theorem 3.2.5: Liouville's Theorem

Let f be entire and bounded. Then f must be constant.

Proof. By the Cauchy estimate, $|f'(z)| \leqslant \sup|f|/r$ for all $r > 0$. Let $r \to \infty$, Done. □

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Proposition 3.2.6

If f is entire and $|f(z)| \leqslant C|z|^n + 1$ for some $C > 0$ and $n \in \mathbb{N}$, then f is a polynomial of degree $\leqslant n$.

Proof. By differentiating and using Cauchy estimate, we have $f^{(n+1)}(z) = 0$. □

3.3 Fundamental Theorem of Algebra & Taylor Series

Theorem 3.3.1: Fundamental Theorem of Algebra

A (complex) polynomial P such that $\deg P \geqslant 1$ has at least one root. (So we can have factorization.)

Proof. Assume P has no zero. Then since $\lim_{z\to\infty} |P(z)| = \infty$, we obtain that $1/P(z)$ is well-defined, bounded, and entire. Therefore P must be constant by Liouville's theorem. □

Definition 3.3.2: Removable Singularity

Let Ω be open. Let $a \in \Omega$ and $f : \Omega\backslash\{a\} \to \mathbb{C}$ be analytic/holomorphic. Then f has a **removal singularity** at a if f can be extended to an analytic function $F : \Omega \to \mathbb{C}$.

Theorem 3.3.3

Let Ω be open and $a \in \Omega$. Let $f : \Omega\backslash\{a\} \to \mathbb{C}$ be analytic. Then f has a removable singularity at a if and only if $\lim_{z\to a}(z-a)f(z) = 0$. An (seemingly weaker but still) equivalent version holds, requiring that f is bounded in a neighborhood of a (excluding a).
(If a is removable, then f must be bounded to ensure F is analytic; conversely if f is bounded then the limit is 0.)

Proof. If a is removable, then

$$\lim_{z\to a} f(z)(z-a) - \lim_{z\to a} F(z)(z-a) = 0$$

Since F is bounded, we must have the second term $= 0$ and so the first term $= 0$.
Conversely, let $r > 0$ be such that $\overline{D_r(a)} \subset \Omega$. Using the more general version of Cauchy's integral theorem for a

disk, we have

$$\int_{\partial D_r(a)} \frac{f(z)-f(\zeta)}{z-\zeta}\, \mathrm{d}\zeta = 0.$$

(Nontrivial claim!) This is because the integrand is analytic in $D_r(a)$ with a singularity satisfying the "nice" condition. There are two singularities: z and a. Define the integrated to be $G(\zeta)$. Then

$$\lim_{\zeta\to a}(\zeta-a)G(\zeta) = \lim_{\zeta\to a}(\zeta-a)\frac{f(z)-f(\zeta)}{z-\zeta} = 0$$

since $\lim_{\zeta\to a}(\zeta-a)f(\zeta)=0$ by assumption, and for fixed z we have $\lim_{\zeta\to a}(\zeta-a)f(z)=0$ as well. Similarly, $\lim_{\zeta\to a}(\zeta-a)G(\zeta)=0$. Therefore the nontrivial claim follows from the generalized Cauchy's Theorem for circles.

This implies that $f(z)=(2\pi i)^{-1}\int_{\partial D_r(a)}\frac{f(\zeta)}{z-\zeta}\,\mathrm{d}\zeta$ for all $z\in D_r(a)\backslash\{a\}$. Now we define

$$F(z) := (2\pi i)^{-1}\int_{\partial D_r(a)} \frac{f(\zeta)}{z-\zeta}\,\mathrm{d}\zeta.$$

Then this is the analytic extension we seek. □

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Theorem 3.3.4: Taylor's Theorem with Analytic Remainder

Let $\Omega\subset\mathbb{C}$ and assume that P is analytic on Ω. Then, for all $n\in\mathbb{N}$, there exists f_n analytic on Ω such that

$$f(z) = \sum_{j=0}^{n-1}\frac{f^{(j)}(a)}{j!}(z-a)^j + f_n(z)(z-a)^n, \qquad z\in\Omega \tag{*}$$

if some $\overline{D_r(a)}\subset\Omega$ (here $a\in\Omega$ is fixed). Moreover, for $z\in D_r(a)$,

$$f_n(z) = \frac{1}{2\pi i}\int_{\partial D_r(a)}\frac{f(\zeta)}{(\zeta-a)^n(\zeta-z)}\,\mathrm{d}\zeta \tag{**}$$

and (*) can be used to obtain an upper bound for the error term.

Proof. We prove by induction. For $n=1$, we have

$$f(z) = f(a) + f_1(z)(z-a).$$

Consider $F(z) := (f(z)-f(a))/(z-a)$. Since F is bounded, a is a removable singularity of F, so we can extend F to a and call the new function f_1.

To prove the inductive step, assume

$$f(z) = \sum_{j=1}^{n-1}\frac{f^{(j)}(a)}{j!}(z-a)^j + f_n(z)(z-a)^n.$$

and consider

$$F(z) = \begin{cases}(f_n(z)-f_n(a))/(z-a) & z\neq a\\ f_n'(x) & z=a.\end{cases}$$

Then F' is analytic in Ω in Ω (bounded assumption plus removable singularity). Then, we have

$$f(z) = \sum_{j=0}^{n-1}\frac{f^{(j)}(a)}{j!}(z-a)^j + f_n(a) = (f_n(a)F(z)(z-a)^n)(z-a)^n$$

and this holds trivially for $z = a$.

We now compute $f_n(a)$ by differentiating it n times:

$$f^{(n)}(z) = o + f_n(a)n! + \frac{\mathrm{d}^n}{\mathrm{d}z^n}(F(z)(z-a)^{n+1}).$$

Therefore$f_n(a) = \dfrac{f^{(n)}(z)}{n!}$. This proves (*).
To prove (**), we note

$$f_n(z) = \frac{1}{2\pi i}\int_{\partial D_r(a)} \frac{f_n(\zeta)}{\zeta - z}\,\mathrm{d}\zeta.$$

From (*) we have

$$f_n(z) = \frac{f(z)}{(z-a)^n} = -\sum_{j=0}^{n-1} \frac{f^{(j)}(0)}{j!(z-a)^{n-j}}$$

To prove (**), it is sufficient to prove that

$$g_n(w) := \int_{\partial D_r(0)} \frac{1}{(\zeta - w)(\zeta - z)}\,\mathrm{d}\zeta = 0.$$

for every $z \in D_r(a)$ fixed and $w \in D_r(a)$. For $n = 1$,

$$g_1(w) = \int_{\partial D_r(a)} \frac{1}{(\zeta - w)(\zeta - z)}\,\mathrm{d}\zeta = \frac{1}{z-w}\int_{\partial D_r(a)}\left(\frac{1}{\zeta - z} - \frac{1}{\zeta - w}\right)\mathrm{d}\zeta = \frac{1}{z-w}\cdot(2\pi i - 2\pi i) = 0.$$

Differentiating g_1, we get

$$g_{n+1}(w) = \frac{g_1^{(n)}(w)}{n!}$$

so that $g_{n+1}(w) = 0$. □

Chapter 4

Singularities of Analytic Functions

4.1 Zeros, Poles, & Unique Continuation

Beginning of Feb.23, 2022

Lemma 4.1.1

Assume that f is analytic in $D_r(a)$ and $f^{(n)}(a) = 0$ for all $n \in \mathbb{N}$ (and 0). Then $f \equiv 0$ in $D_r(a)$.

Intuition: if we know existence of Taylor series then this is obvious. But we don't.

Proof. WLOG assume f is analytic in $\overline{D_r(a)}$. Let $M := \sup_{\overline{D_r(a)}} |f|$. By Taylor's theorem, for every n, there exists f_n analytic in $D_r(a)$ with

$$f(z) = f_n(z)(z-a)^n + \sum_{j=0}^{n-1} \frac{f^{(j)}(a)}{j!}(z-a)^j = f_n(z)(z-a)^n.$$

Also,

$$f_n(z) = \frac{1}{2\pi i} \int_{\partial D_r(a)} \frac{f(\zeta)}{(\zeta-a)^n(\zeta-z)}\, d\zeta.$$

We estimate

$$\begin{aligned} |f_n(z)| &\leqslant \frac{2\pi r \sup|f|}{2\pi r^n \cdot (r-|z-a|)} = \frac{M}{r^{n-1}(r-|z-a|)} \\ &\leqslant \frac{M|z-a|^n}{r^{n-1}(r-|z-a|)}. \end{aligned}$$

Since $|z-a| < r$, as $n \to \infty$ we must have $|f_n(z)| \to 0$. Thus $f \equiv 0$. $\square$

Theorem 4.1.2: Taylor Expansion

Let f be analytic in $\Omega \subset \mathbb{C}$ and let $a \in \Omega$, $r > 0$ such that $\overline{D_r(a)} \subset \Omega$. Then

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n \qquad \text{for all } z \in D_r(a).$$

The expression for "error" is $R(z) = f_n(z)(z-a)^n$ as mentioned previously. This converges to 0 exponentially. The rate of convergence of the power series is at least

$$R = \sup\{r > 0 : \overline{D_r(a)} \subset \Omega\}.$$

Theorem 4.1.3: Unique Continuation

Assume f is analytic in $\Omega \subset \mathbb{C}$ which is open and *connected* and satisfies

$$f^{(n)}(a) = 0 \qquad \text{for all } n \in \mathbb{N} \cup \{0\} \tag{*}$$

for *some* $a \in \Omega$. Then $f \equiv 0$ in Ω.
Alternatively if f vanishes on an nonempty open set then f vanishes everywhere.

Proof. The second assertion is a consequence of the first, so we only prove the one associated with (*).
Let

$$A := \{z \in \Omega : f^{(n)}(z) = 0 \text{ for all } n \in \mathbb{N} \cup \{0\}\}.$$

This is nonempty because of (*). It is open by the previous lemma. It is also closed because of continuity of $f^{(n)}$. Therefore, with Ω connected, the only possibility is if $A = \Omega$. This completes the proof. □

Remark. For $C^\infty(\mathbb{R})$, the statement is not true in general! We can consider bump functions like

$$f(x) := \begin{cases} e^{-1/x^2} & x > 0 \\ 0 & x \leqslant 0 \end{cases}$$

where $f^{(n)}(0) = 0$ for all n.

Example 4.1.4. Suppose f is analytic at 0 and

$$f^{(n)}(0) = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases}$$

Then $(f - z)^n = 0$ for all $n \in \mathbb{N} \cup \{0\}$. That is, $f \equiv z$. More generally, if f, g are analytic in Ω and $f^{(n)}(a) = g^{(n)}(a)$ for some $a \in \Omega$ and all $n \in \mathbb{N} \cup \{0\}$, then $f = g$.

Definition 4.1.5

Assume f is analytic in Ω and is not identically zero. Then, for $a \in \Omega$ with $f(a) = 0$, we define

$$\min\{n \in \mathbb{N} : f^{(n)}(a) \neq 0\}$$

to be the **order of vanishing** or the **order of the zero** of a. If $f(a) \neq 0$ we define the order to be 0. We use the notation $\operatorname{ord}_z f = n$.

Assume f is analytic in Ω and $\text{ord}_z f = n$ for some $z \in \Omega$. Then by Taylor expansion, there exists f_n analytic in Ω with

$$f(z) = (z-a)^n f_n(z)$$

and $f_n(a) \neq 0$. That is, we can factor out zeros!

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Theorem 4.1.6

If f is analytic and is not identically 0 in $\Omega \subset \mathbb{C}$, then $\{f = 0\}$ (the zero set) does not have an accumulation point in Ω.
Consequently, if f, g are analytic in Ω and $f = g$ agree on a set with an accumulation point in Ω, then $f = g$. (If the accumulation point is on the boundary, it does not count.)

Example 4.1.7. If f, g are analytic in $\Omega \supset \mathbb{R}$ (connected) and $f \equiv g$ on $\mathbb{R}$, then $f \equiv g$ on Ω.

Proof. Assume that $a \in \Omega$ is a zero of order n. Then there exists g analytic in Ω with $f(z) = (z-a)^n g(z)$ and $g(a) \neq 0$. Therefore by continuity g is locally nonzero around a, so each zero is an isolated zero! □

We temporarily fix the following assumptions:
Let $a \in \mathbb{C}$ and $r > 0$. Fix an analytic function f in $\Omega \backslash \{a\}$ where $a \in \Omega$. Assume that $\overline{D_r(a)} \subset \Omega$.
We call a an **isolated singularity**.

Definition 4.1.8: Types of Singularity

The singularity of a is either

(1) a **removable singularity**, if f is bounded in $D_\rho(a) \backslash \{a\}$ for some $\rho \in (0, r)$, as stated before;

(2) a **pole**, if $\lim_{z \to a} f(z) = \infty$ (we'll expand later); or

(3) an **essential singularity** otherwise.

We say $\lim_{z \to a} f(z) = \infty$ if, for all $M > 0$, there exists $r > 0$ such that $f(z) > M$ for $z \in D_r(a) \backslash \{a\}$.
If a is a pole, it is reasonable to define $f(a) = \infty$, and it turns out that f is analytic with values in $\mathbb{C}_\infty$.

Theorem 4.1.9

Assume that a is a pole of f. Then there exists a unique $n \in \mathbb{N}$ and g analytic in Ω such that $g(a) \neq 0$ and

$$f(z) = \frac{g(z)}{(z-a)^n} \qquad \text{for all } z \in \Omega \backslash \{a\}.$$

If we define $f(a) = \infty$ then the above holds for all $z \in \Omega$. (Note the connection to rational functions here.)

Studying poles, in some sense, is equivalent to studying zeros of $1/f$.

Proof. The function $1/f(z)$ is analytic in $\Omega \backslash \{f = 0\}$, where Ω is open and $\{f = 0\}$ discrete. Then $1/f(z)$ has a removable singularity at a and its extension at a vanishes at a. Therefore there exists a unique $n \in \mathbb{N}$ such that

$$\frac{1}{f(z)} = (z-a)^n G(z)$$

for some G analytic on $\Omega \backslash \{f = 0\}$ with $G(a) \neq 0$. Therefore $f(z) = \dfrac{1/G(z)}{(z-a)^n}$ for the same domain. Let $g(z) := 1/G(z)$ on $\Omega \backslash \{f = 0\}$.

Since $g(a) \neq 0$ follows trivially, our final step is to show that we can extend g to all of Ω; that is, $\{f = 0\}$ are all removable singularities of g; that is, we check g is bounded locally around each point in $\{f = 0\}$, but this follows from the fact that

$$g(z) = f(z)(z-a)^n$$

where $f(z)$ and $(z-a)^n$ are both locally bounded around zeros of f. (For $b \in \{f = 0\}$, we have $\lim_{z \to b} g(z) = \lim_{z \to b} f(z)(z-a)^n = f(b)(b-a)^n$ which is still removable.) □

Beginning of Feb.29, 2022

Theorem 4.1.10

Let a be a pole and $n \in \mathbb{N}$ its order. Then there exist unique nonzero $b_1, b_2, \dots, b_n \in \mathbb{C}$ and φ analytic in $\Omega \backslash \{a\}$ such that

$$f(z) = \frac{b_n}{(z-a)^n} + \dots + \frac{b_1}{z-a} + \varphi(z) \qquad \text{for all } z \in \Omega \backslash \{a\}.$$

We all all but the last term ($\varphi(z)$) the *singular part* at a.

Proof. By Taylor expansion, since a is a removable singularity for $(z-a)^n f(z)$, we have

$$f(z)(z-a)^n = b_n + b_{n-1}(z-a) + \dots + b_1(z-a)^{n-1} + \varphi(z)(z-a)^n$$

for every $z \in \Omega \backslash \{a\}$. Dividing by $(z-a)^n$ gives the claim. □

Theorem 4.1.11: Casorati-Weierstrass Theorem

Assume that a is an essential singularity. Then for every $\rho \in (0,1)$, we have

$$\overline{f(D_\rho \backslash \{a\})} = \mathbb{C}.$$

Example 4.1.12. $e^{1/z}$ has an essential singularity at 0. It does not take the value 0 or ∞.

Theorem 4.1.13: Great Picard's Theorem

For a an essential singularity and for every $\rho \in (0,1)$, we have

$$f(D_\rho(a) \backslash \{a\}) = \text{ either } \mathbb{C} \text{ or } \mathbb{C} \backslash \{b\} \text{ for some } b \in \mathbb{C}.$$

Theorem 4.1.14: Little Picard's Theorem

If f is entire and nonconstant, then $f(\mathbb{C})$ is either $\mathbb{C}$ or $\mathbb{C}\setminus\{b\}$ for some b.

Proof of Casorati-Weierstrass. Assume the closure statement does not hold. Then there exists $g \in (0,1)$, $A \in \mathbb{C}$, and $\delta > 0$, such that

$$|f(z) - A| \geqslant \delta \qquad \text{for all } z \in D_\rho(a)\setminus\{a\}.$$

This implies that $\lim_{z\to a} \frac{|f(z)-A|}{|z-a|} = \infty$. Therefore the function $(f(z)-A)/(z-a)$ has a pole at a. Therefore there exists $n \in \mathbb{N}$ and g analytic on $D_\rho(a)$ such that $g(a) \neq 0$ and

$$\frac{f(z)-A}{z-a} = \frac{g(z)}{(z-a)^n} \qquad \text{for all } z \in D_\rho(a)\setminus\{a\}.$$

Therefore

$$f(z) = A + \frac{g(z)}{(z-a)^{n-1}} \qquad \text{for all } z \in D_r(a)\setminus\{a\}$$

so f is either a pole (if $n \geqslant 2$) or a removable singularity (if $n = 1$). □

Isolated Singularity at ∞

Assume that f is analytic in Ω and $D_r^c \subset \Omega$ for some $r > 0$. Now consider the function

$$g(z) := f(1/z).$$

Then f is a removable singularity / pole / essential singularity at ∞ if and only if g has the same thing at 0.

Example 4.1.15. Let P be a polynomial. Then ∞ is a pole if $\deg P \geqslant 1$.

Beginning of March 2, 2022

4.2 Local Mapping Properties

There are a number of ways to count zeroes in regions.

Theorem 4.2.1

Let $z_1, z_2, \dots$ be all the zeros of f that is not identically zero in $D_r(a)$. Let γ be a closed curve in D which does not pass through any zeros. Then

$$\sum n(\gamma, z_i) = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z.$$

This implies that almost all zeros have zero index.

Since analytic functions have discrete zeros, if f (nonzero) is analytic in a connected Ω and $K \subset \Omega$ is compact, then f can only have finitely many zeros in K. We will use his fact in the proof.

Proof. By reducing the radius we may assume that there are only finitely many zeros, $z_1, \ldots, z_n$ of f, repeated according to their multiplicities. (Use the compact argument above.)
Then there exists a nonzero g in D such that

$$f(z) = (z - z_1)(z - z_2)\ldots(z - z_n)g(z).$$

We define the *logarithmic derivative* by applying the Leibniz rule:

$$f'(z) = \sum_{i=1}^{n} g(z) \prod_{j \neq i} (z - z_j) \implies \frac{f'(z)}{f(z)} = \frac{1}{z - z_1} + \ldots + \frac{1}{z - z_n} + \frac{g'(z)}{g(z)}.$$

Integrating over γ, we have

$$\int_\gamma \frac{f'(z)}{f(z)}\, \mathrm{d}z = 2\pi i \sum_{i=1}^{n} n(\gamma, z_n) + 0.$$

□

Using chance of variable $w = f(z)$, we have

$$\int_\gamma \frac{f'(z)}{f(z)} = \int_{f \circ \gamma} \frac{\mathrm{d}w}{w}.$$

Then the number of zeros in γ is simply the index of 0 in $f \circ \gamma$. That is,

Corollary 4.2.2

Under the assumptions of the previous theorem, we have

$$\sum n(\gamma, z_j) = n(\Gamma, 0),$$

where $\Gamma = f \circ \gamma$. In particular, if γ is a simple closed curve which does not intersect itself, then the total number of zeros inside γ (with multiplicities counted repeatedly) equals the index of $f \circ \gamma$ around 0.

This can be applied to $f(z) - a$ for any $a \in \mathbb{C}$. We obtain

$$\sum_j n(r, z_j(a)) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z) - a}\, \mathrm{d}a$$

where $z_j(a)$ are the zeros of $f - a$. After changing variable we see that the RHS is also the index of $f \circ \gamma$ around a.
Upshot. The winding number does not change if we perturb a (i.e., if Δa is small).

Theorem 4.2.3

Assume that f is analytic at z_0 and $f(z) - w_0$ has a zero of order n at z_0. Then for $\delta > 0$ sufficiently small, there exists $\epsilon > 0$ such that

$$f(z) - a \text{ has } n \text{ distinct simple zeros in } D_\epsilon(z_0) \qquad \text{for all } a \in D_\delta(w_0) \backslash \{w_0\}.$$

Beginning of March 4, 2022

Proof. Let $\epsilon_0 > 0$ be small. Choose δ small so that $f', f - w_0$ has no zeros in $D_\delta(z_0) \backslash \{z_0\}$. Since zeros are isolated such action is positive. Also assume $\overline{D_\delta(z_0)} \subset \Omega$. Consider $r_\delta := \partial D_r(z_0)$ and denote Γ_δ as the composition $f \circ r_\delta$.

Choose ϵ small so that $\overline{D_\epsilon(w_0)}$ is in one component, i.e., $\overline{D_\epsilon(w_0)} \cap \{\Gamma\} = \varnothing$.
Let $a \in D_\epsilon(w_0)\backslash\{w_0\}$. Then a has the same index number as w_0, and $f(z)-a$ has n zeros inside $D_\epsilon(w_0)$ (counting multiplicities) as shown in the index theorem above. By construction of ϵ_0, f' has no zeros in $D_\delta(z_0)\backslash\{z_0\}$, so the multiplicities can only be one. □

Corollary 4.2.4

Let f be analytic, nonconstant, and Ω connected. Then f maps every open set to an open set. We say such a mapping is **open**.

Corollary 4.2.5

Let f be analytic at z_0 with $f'(z_0) \neq 0$. Then there exists an open neighborhood of z_0 which is mapped conformally and homeomorphically to a neighborhood of $f(z_0)$.

Proof. Let $\delta > 0$ be sufficiently small. Then we can reduce it (if necessary) so that $f'(z) \neq 0$ in $D_\delta(z_0)$. Then use the previous theorem, as f maps $f^{-1}(D_\epsilon(w_0))$ conformally and homeomorphically to $D_\epsilon(w_0)$. □

Higher Order Multiplicities

Intuitively: if a is a zero of multiplicity n, then f behaves like z^n near a.

Beginning of March 7, 2022

More formally, assume that the multiplicity of $F(z) - f(z_0)$ at z_0 is $n \geqslant 2$. Denote $w_0 := f(z_0)$. Then there exists g such that $f_n - w_0 = (z - z_0)^n g(z)$, $g(z) \neq 0$.
By continuity, there exists $\epsilon > 0$ such that $|g(z) - g(z_0)| < |g(z_0)|$ for all $|z - z_0| < \epsilon$. Thus we can define $\log$ on the set $D_{|g(z_0)|}(g(z_0))$, and thus exists $h(z) := (g(z))^{1/n}$ or $\exp(\log g(z)/n)$, and

$$f(z) - w_0 = (z - z_0)^n h(z)^n =: \zeta(z)^n$$

where $\zeta(z) = (z - z_0)h(z)$. Note that

$$\zeta'(z) = h(z) + (z - z_0)h'(z) \implies \zeta'(z_0) = h(z_0) \neq 0$$

since $g(z_0) \neq 0$. By the previous corollary, $\zeta(z)$ is a local conformal homeomorphism. Hence f can be written as a composition $z \mapsto \zeta(z)$ and then $\zeta(z) \mapsto (\zeta(z))^n$.

4.3 Maximum Principle

Theorem 4.3.1

Assume that f is nonconstant and analytic in $\Omega \subset \mathbb{C}$ (open and connected). Then $|f(z)|$ does *not* attain a local maximum in Ω.
The converse is clearly false, as we have analytic functions that vanish at a point. However if we exclude

this case then the theorem holds by the same reasoning below.

Proof. Let $w_0 = f(z_0)$ where $z_0 \in \Omega$. Since f is open, it maps open neighborhoods of z_0 to open neighborhoods of w_0. Thus f cannot attain a local maximum at z_0 (in an open neighborhood, there always exists some point with larger modulus). $\square$

Beginning of March 9, 2022

Corollary 4.3.2

Assume that Ω is open, bounded, and connect. Assume f nonconstant is analytic in Ω and continuous on $\overline{\Omega}$. Then f achieves its maximum only on the boundary. (It has to attain max on $\overline{\Omega}$ by compactness and continuity, whereas maximum principle prohibits such maximum to be in Ω.)

4.4 Schwarz Lemma

The Schwarz lemma can be considered as an enforcement of the maximal principle for a disk.

Theorem 4.4.1: Schwarz Lemma

Assume that f is analytic in $\mathbb{D}$ and satisfies $|f(z)| \leqslant 1, f(0) = 0$. Then $|f(z)| \leqslant |z|$ and $|f'(0)| \leqslant 1$.
If in addition there exists $z \in \mathbb{D}$ such that $|f(z)| = |z|$ *or* if $|f'(0)| = 1$, then there exists $c \in \mathbb{C}$ with $|c| = 1$ such that $f(z) = cz$.

Proof. Consider the analytic function

$$g(z) = \begin{cases} f(z)/z & z \in \mathbb{D} \backslash \{0\} \\ f'(0) & z = 0 \end{cases}$$

which is analytic (since 0 is a removable singularity or alternatively $f(z) = zg(z)$). Then

$$|g(z)| = |f(z)| \leqslant 1 \qquad \text{for all } z \in \partial\mathbb{D}$$

and by maximum principle $|g(z)| \leqslant 1$ so $|f(z)| \leqslant |z|$ in the disk.
The second claim follows since $|g|$ cannot have an interior maximum 1 unless it's constant. $\square$

Corollary 4.4.2: Schwarz-Pick

If $f : \mathbb{D} \to \mathbb{C}$ is analytic and $|f(z)| \leqslant 1$ for all $|z| \leqslant 1$ and $f(z_0) = w_0$, then

$$\left|\frac{f(z) - f(z_0)}{1 - \overline{f(z_0)}f(z)}\right| \leqslant \left|\frac{z - z_0}{1 - \overline{z_0}z}\right|$$

and

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leqslant \frac{1}{1 - |z|^2}.$$

For the proof, we use $S : w \mapsto (w - w_0)/(1 - \overline{w_0}w)$, $T : z \mapsto (z - z_0)/(1 - \overline{z_0}z)$, where $w_0 = f(z_0)$. Then apply Schwarz lemma to $S \circ f \circ T^{-1}$.

Remark. This theroem says that analytic f with $|f| \leqslant 1$ on $\mathbb{D}$ reduces the *Poincare* distance

$$d(z, z_0) = \left|\frac{z - z_0}{1 - \overline{z_0}z}\right|.$$

If $|f| \leqslant 1$ then

$$d(f(z_0, f(z_2)) \leqslant d(z_1, z_2).$$

4.5 General Form of the Cauchy Formula

Let $\gamma_1, \ldots, \gamma_n$ be piecewise smooth curves. We define

$$\int_{\sum \gamma_i} f \, \mathrm{d}z = \sum_{i=1}^{n} \int_{\gamma_i} f \, \mathrm{d}z.$$

The "sum" $\gamma := \sum \gamma_i$ is called a **chain**. Two chains are considered the same if they lead to the same line integral for *every* continuous f.

We can represent every chain as $\gamma = a_1\gamma_1 + \ldots + a_n\gamma_n$

Beginning of March 21, 2022

(Missed March 11: definition of a cycle and simply connected domains. Theorem: a region is simply connected iff the index of every curve w.r.t. every point in the complement is zero.)

Index Zero $\Rightarrow$ Simply Connected. Suppose Ω is not simply connected; that is, there exist A, B disjoint and closed such that $\Omega^c = A \cup B$.

Let A be the bounded one. Let δ be the distance between A and B, and cover the plane with a net of squares of side length $\leqslant \delta/4$ (so that B is some squares away from A). If necessary, we slightly adjust the net to ensure that at some point $a \in A$ is not on any edge. Consider

$$\gamma := \sum_{q \in \mathcal{A}} \partial q$$

where $\mathcal{A}$ is the net. Since overlapping edges cancel, γ is the smallest "zigzag" cycle containing γ. (Each boundary is a cycle so the sum also is.) Furthermore, $A \cap \{\gamma\} = \varnothing$. (Otherwise a point $a \in A \cap \{\gamma\}$ needs to belong to either two or four squares, and it will get cancelled.) Note that $n(\gamma, a) = 1$. This is because $a \in q_0$ for some $q_0 \in A$, and so $n(\partial q_0, a) = 1$ and $n(\partial q, a) = 0$ for other q's. Adding them up gives $n(\gamma, a) = 1$.

For points in A that lie on the mesh boundaries, we can "merge" two or four adjacent squares and see that the indices are still 1. □

The notion of singly connected domains is useful for multiply connected domains, which we will discuss later.

For (finitely many) multiply connected regions, we can pick δ sufficiently small such that each bounded component of Ω^c is separated by the δ-mesh. For (infinitely many) multiply connected region this might fail: for example

$\mathbb{D}\setminus\bigcup_{n\geqslant 1}\{1/n\}$. Then $n(\gamma,a)=0$ for all a in the unbounded component of Ω^c and $n(\gamma,a)=1$ for all a in the union of bounded components of Ω^c.

If Ω is bounded, we can do a similar mesh thing for Ω (to obtain a large zigzag γ contained by Ω, so it is closed and is in Ω) and get a "zigzag" boundary path γ for Ω.

Proposition 4.5.1

Assume Ω is open and connected but *not* simply connected. Then there exists f analytic in Ω and a cycle γ such that

$$\int_\gamma f(z)\,\mathrm{d}z \neq 0.$$

Proof. By the previous theorem there exist a cycle γ and a point $a\in\Omega^c$ such that $n(\gamma,a)=1$. Hence

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{z-a}\,\mathrm{d}z = 1.$$

Done: let $f(z) := 1/(z-a)$. □

Definition 4.5.2

We say $\gamma\in\mathcal{C}_0$ is **homologous to 0** with respect to Ω (we write $\gamma\sim 0 \pmod{\Omega}$) if $n(\gamma,a)=0$ for all $a\in\Omega^c$. In particular, if Ω is simply connected, $\gamma\sim 0 \pmod{\Omega}$ for all $\gamma\in\mathcal{C}_0$.

Theorem 4.5.3: Cauchy Theorem, General Form

Assume f is analytic in Ω connected and γ is homologous to 0. Then

$$\int_\gamma f(z)\,\mathrm{d}z = 0.$$

Before now, Cauchy theorem required the domain to have the property that any z_1, z_2 can be connected via a zigzag path.

Corollary 4.5.4: Cauchy Theorem for Simply Connected Region

Let Ω be simply connected and f analytic in Ω. Then

$$\int_\gamma f(z)\,\mathrm{d}z = 0 \qquad \text{for all } \gamma\in\mathcal{C}_0.$$

Corollary 4.5.5

Let Ω be simply connected. Let f be analytic in Ω such that $f(z)\neq 0$ for all $z\in Omega$. Then one can define $\log f(z)$, i.e., there exists a g analytic with

$$e^{g(z)} = f(z).$$

Consequently, we can define $f(z)^\alpha$ for $\alpha\in\mathbb{R}$.

Beginning of March 23, 2022

Proof. We apply the Cauchy theorem to f'/f. There exists F such that $F' = f'/f$. Then

$$(f(z)\exp(-F(z))' = f'e^{-F} - \frac{f'}{f}fe^{-F} = 0.$$

Therefore there exists $a \in \mathbb{C}$ such that $f(z)e^{-F(z)} = a\alpha$.
Fix $z_0 \in \Omega$. Then $\arg f(z_0) \in [0, 2\pi)$. Then

$$f(z)e^{-F(z)} = f(z_0)e^{-F(z_0)} = \exp(\log|f(z_0)| + i\arg f(z_0))\exp(-F(z_0))$$

so

$$f(z) = \exp(F(z) + \log|f(z_0)| + i\arg f(z_0) - F(z_0))$$

and we are done. □

Proof of Cauchy Theorem. Assume Ω is bounded (otherwise since γ is bounded we can shrink it) and $\gamma \sim 0 \pmod{\Omega}$. We cover the plane with squares by size δ. Let $\mathcal{A}$ be the set of all squares and let $\mathcal{A}_0$ be the set contained in $\overline{\Omega}$. Define

$$\Omega_\delta := \text{int} \bigcup_{Q \in \mathcal{A}_0} \overline{Q}.$$

(That is, Ω_δ is the collection of δ-squares entirely contained in Ω.)
If δ is sufficiently small (smaller than the distance between γ and Ω^c), then γ is contained in Ω_δ. Define

$$\Gamma_\delta := \sum_{Q \in \mathcal{A}_0} \partial Q$$

so that intuitively $n(\gamma, \zeta) = 0$ for $\zeta \in \Gamma_\delta$ (since γ is away from Γ_δ).
To see this, if $z \in Q_0 \in \mathcal{A}_0$, then

$$\frac{1}{2\pi i}\int_{\partial Q} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = \begin{cases} f(z) & \text{if } Q = Q_0 \\ 0 & \text{otherwise} \end{cases}.$$

Therefore

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma_\delta} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta \qquad \text{for all } z \in \Omega_\delta$$

so

$$\int_\gamma f(z)\,\mathrm{d}z = \int_\gamma \frac{1}{2\pi i}\int_{\Gamma_\delta} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta\,\mathrm{d}z = \int_{\Gamma_\delta} f(\zeta)\,\mathrm{d}\zeta\left(\frac{1}{2\pi i}\int_\gamma \frac{1}{\zeta - z}\,\mathrm{d}z\right) = \int_{\Gamma_\delta} f(\zeta)\,\mathrm{d}\zeta \cdot 0 = 0.$$

□

Definition 4.5.6

A differential $p\mathrm{d}x + q\mathrm{d}y$ (p, q continuous) is **locally exact** if it is exact in a bounded neighborhood of every point in Ω.

Beginning of March 25, 2022

Proposition 4.5.7

$p\,\mathrm{d}x + q\,\mathrm{d}y$ is locally exact if and only if

$$\int_{\partial R} p\,\mathrm{d}x + q\,\mathrm{d}y = 0$$

for every rectangle R with sides parallel to the axes and $\overline{R} \subset \Omega$.

Proof. For $\Leftarrow$, use R as the neighborhood and the differential is locally exact.
For $\Rightarrow$, subdivide R into sufficiently small. Use local exactness *and compactness* to prove the claim. □

Theorem 4.5.8

If $p\,\mathrm{d}x + q\,\mathrm{d}y$ is locally exact in Ω, then

$$\int_{\gamma} p\,\mathrm{d}x + q\,\mathrm{d}y = 0$$

for every cycle γ such that $\gamma \sim 0 \pmod{\Omega}$.
This generalizes Cauchy theorem because if f is analytic then $f(z)\,\mathrm{d}z$ is locally exact.

Proof. We can replace γ with a polygonal curve with edges parallel to the axes. Note that γ does not contain a inner curve or otherwise the original γ is not homological to 0. We extend every segment into infinite lines so that the plane and in particular γ is partitioned into various rectangles.
Let R_1 be the collection of all the finite rectangles and R_2 the infinite ones (half strips). For each $R \in \mathbb{R}1 \cup \mathbb{R}2$, choose a point a_R in the interior. Let

$$\sigma_0 := \sum_{R \in R_1 \cup R_2} n(\gamma, a_R)\partial R.$$

We will show that $\gamma = \sigma_0$. Easy computations show that

$$n(\gamma_0, a_R) = \begin{cases} n(\gamma, a_R) & \text{for all } R \in R_1 \\ n(\gamma, a_R) = 0 & \text{for all } R \in R_2. \end{cases}$$

Therefore $n(\gamma - \sigma_0, a_R) = 0$ for all $R \in R_1 \cup R_2$, and we can replace a_R by any point not on the edges.
Let σ be any common side between two finite rectangles R' and R'', where R' is on the left side of σ relative to its orientation.
Assume that the reduced representation of $\gamma - \sigma_0$ has some $c\sigma$ where σ is an cycle. Consider

$$\sigma - \sigma_0 - c\partial R'$$

which is a cycle. This cycle does not contain σ. Then

$$n(\sigma - \sigma_0 - c\partial R', a_{R'}) = n(\sigma - \sigma_0 - c\partial R'', a_{R''})$$

since the σ in-between is no longer present and the winding number must be constant when moving inside the same rectangle. However, the LHS is $-c$ and the RHS is 0. Similar proof for infinite strips. Therefore $c = 0$. Hence

$$\gamma = \sum_{R \in R_1} n(\sigma, a_R)\partial R.$$

This is because the expression for $\gamma - \sigma_0$ contains no edge. It remains to show that every rectangle $R \in R_1$ such that $n(\gamma, a_R)$ has the property $\overline{R} \subset \Omega$. This holds since $\gamma \sim 0 \pmod{\Omega}$: if R has a point not in Ω, then for that point $n(\gamma, a_R) = 0$. □

Beginning of March 28, 2022

4.6 Multiply Connected Domain

Definition 4.6.1

Ω has **connectivity** $n \in \mathbb{N}$ if $\mathbb{C}_\infty \backslash \Omega$ has n connected components.

For example, $\Omega := \mathbb{C}$ has connectivity 1. $\mathbb{C}\backslash\{0\}$ has connectivity 2: the complement in $\mathbb{C}_\infty$ has two components, one $\{0\}$, the other $\{\infty\}$.

Intuitively, if we let $A_1, \ldots, A_n$ be the connected components of $\mathbb{C}_\infty\backslash\Omega$, we can construct $\gamma_1, \ldots, \gamma_{n-1}$ such that $n(\gamma_i, a) = 1$ for $a \in A_i \in \mathbb{C}_\infty\backslash\Omega$ and $n(\gamma_i, a) = 0$ for $a \in \Omega^c\backslash A_i$.

Let $\gamma \in C_0$ (cycles) and let $c_i := n(\gamma, a)$ for $a \in A_i$. Then

$$\gamma \sim c_1\gamma_1 + \ldots + c_{n-1}\gamma_{n-1} \pmod{\Omega}$$

since if $a \in A_i$ for some fixed $i \in \{1, \ldots, n-1\}$ then

$$n(\gamma - \sum_{j \neq i} c_j\gamma_j, a) = n(\gamma, a) - c_i n(\gamma_i, a) = 0.$$

Then, let $a \in A_i$ for some i, and

$$n(c_1\gamma_1 + \ldots + c_{n-1}\gamma_{n-1}, a) = 0$$

We call $\gamma_1, \ldots, \gamma_{n-1}$ a **homology basis** for Ω.

Application

We would like to compute $\int_\gamma f \, \mathrm{d}z$ using $\int_{\gamma_i} f \, \mathrm{d}z$.

If $\gamma \in C_0$, the there exist unique $c_1, \ldots, c_{n-1} \in \mathbb{Z}$ such that

$$\gamma \sim c_1\gamma_1 + \ldots + c_{n-1}\gamma_{n-1} \pmod{\Omega}.$$

Then

$$\int_\gamma f \, \mathrm{d}z = c_1 \int_{\gamma_1} f \, \mathrm{d}z + \ldots + c_{n-1} \int_{\gamma_{n-1}} f \, \mathrm{d}z.$$

The integrals

$$P_i := \int_{\gamma_i} f \, \mathrm{d}z$$

are called **modulus of periodicity** or **periods**.

Proposition 4.6.2

If all periods of f vanish, e.g., if Ω is simply connected, then $\int_\gamma f \, \mathrm{d}z = 0$ for all cycles γ and thus f has a primitive. This is a version of Cauchy's theorem for a n-connected domain.

Example 4.6.3. Let $\Omega := \mathbb{D}\backslash\{0\}$. Then $\gamma_1 = \partial D_\delta(0)$ for any $\delta \in (0,1)$.

4.7 Residues

The calculus of residues is based on the following theorem.

Theorem 4.7.1

Assume f is analytic in $D_r(a)\backslash\{a\}$. Then there exists a unique $R \in \mathbb{C}$, called the **residue** of f at a, $\text{res}_a f$, and an analytic function F in $D_r(a)\backslash\{a\}$, such that

$$f(z) - \frac{R}{z-a} = F'(z).$$

In particular,

$$\int_\gamma f(z)\,\mathrm{d}z = 2\pi i R n(\gamma, a).$$

Proof. Let $\gamma_1 = \partial D_\rho(a)$ where $\rho \in (0, r)$. Let

$$R := \frac{1}{2\pi i}\int_{\gamma_1} f(z)\,\mathrm{d}x.$$

The function

$$g(z) = f(z) - \frac{R}{z-a}$$

satisfies

$$\int_{\gamma_1} g(z)\,\mathrm{d}z = \int_{\gamma_1} f(z)\,\mathrm{d}z - \int_{\gamma_1}\frac{R}{z-a}\,\mathrm{d}z = 2\pi i R - 2\pi i R.$$

Therefore $\int_\gamma g(z)\,\mathrm{d}z = 0$ for all $\gamma \in C_0$. Therefore g has a primitive. □

Beginning of March 30, 2022

Theorem 4.7.2: Residue Theorem

Assume that f is analytic except on a discrete set $A = \{a_1, a_2, \ldots\} \subset \Omega$ and that f is analytic in $\Omega\backslash A$. Then

$$\int_\gamma f\,\mathrm{d}z = 2\pi i \sum_j n(\gamma, a_j)\text{Res}(f, a_j)$$

for every $\gamma \in C_0$ not intersecting A and $\gamma \sim 0$. Since A is discrete, we have $n(\gamma, a_j) = 0$ for all but finitely many.

Proof. WLOG assume $A = \{a_1, \ldots, a_n\}$ is finite (for we can always throw away those with index 0). Let $\gamma_1, \ldots, \gamma_n$ be sufficiently small circles so that the disks are separated. Also assume that these disks are entirely in Ω. Then

$$g(z) := f - \sum_{j=1}^n \frac{\text{Res}(f, a_j)}{z - a_j}$$

is periodic. So

$$\int_\gamma f - \sum_{j=1}^{n} \frac{\text{Res}(f, a_j)}{z_j}\, \mathrm{d}z = 2\pi i \sum_j \text{Res}(f, a_j) n(\gamma, a_j).$$

□

Computing Residues at Poles

If f has a pole of order n, then

$$f(z) = \frac{g(z)}{(z-a)^n}$$

where g is analytic and $g(0) \neq 0$. By Taylor's theorem,

$$f(z) = g(a) + (z-a)g'(a) + \dots + (z-a)^{n-1} \cdot \frac{g^{n-1}(a)}{(n-1)!} + (z-a)^n h(z)$$

where $h(z)$ is also analytic.
Then

$$\text{Res}(f, a) = \frac{f^{(n-1)}(a)}{(n-1)!}.$$

Inductively we have

$$\begin{aligned} g(a) &= \lim_{z\to a} f(z)(z-a)^n \\ g'(a) &= \lim_{z\to a} \left(f(z)(z-a)^{n-1} - \frac{g(a)}{z-a} \right) \\ \frac{g^{(n-1)}(a)}{(n-1)!} &= \lim_{z\to a} \left(f(z)(z-a) - \frac{g(a)}{(z-a)^{(n-1)}} - \frac{g'(a)/1!}{(z-a)^{n-2}} - \dots - \frac{g^{(n-1)}(a)/(n-2)!}{z-a} \right). \end{aligned}$$

Example 4.7.3. Compute $\int_0^\infty \frac{\sin x}{x}\, \mathrm{d}x$ using residue theorem. Note that

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

and e^{iz} is small in the upper plane and e^{-iz} small in the lower plane. We use

$$\frac{\sin x}{x} = \mathfrak{Im} \frac{e^{ix}}{x}$$

which has a pole. Let $R > \epsilon > 0$ and we consider the upper ring induct by the disks with radii R and ϵ. Call the upper semicircle of ϵ r_ϵ and the one by R, r_R. Then by the residue theorem / Cauchy theorem,

$$\int_{-R}^{-\epsilon} \frac{e^{iz}}{z}\, \mathrm{d}z + \int_{r_\epsilon} \frac{e^{iz}}{z}\, \mathrm{d}z + \int_\epsilon^R \frac{e^{iz}}{z}\, \mathrm{d}z + \int_{r_R} \frac{e^{iz}}{z}\, \mathrm{d}z = 0.$$

We claim that

$$\lim_{\epsilon\to 0} \int_{\gamma_\epsilon} \frac{e^{iz}}{z}\, \mathrm{d}z = -\frac{1}{2} 2\pi i \text{Res}(e^{iz}/z, 0) = -\pi i.$$

To see this, note that $|(e^{iz} - 1)/z|$ is bounded by some $M > 0$ for $|z| \in (0, 1)$. Therefore

$$\lim_{\epsilon\to 0} \int_{r_\epsilon} \frac{e^{iz} - 1}{z}\, \mathrm{d}z = \lim 2\pi\epsilon = 0.$$

Therefore

$$\lim_{\epsilon\to 0} \int_{r_\epsilon} \frac{e^{iz}}{z}\, \mathrm{d}z = \lim_{\epsilon\to 0} \int_{r_\epsilon} \frac{1}{z}\, \mathrm{d}z = -\pi i.$$

We claim

$$\lim_{R\to\infty}\int_{r_R}\frac{e^{iz}}{z} = \mathrm{d}z = 0.$$

To this end, partition r_R into three parts, $r_R^{(1)}, r_R^{(2)}, r_R^{(3)}$ so that $r_R^{(2)}$ is the part with $\Im\mathfrak{m} \geqslant \sqrt{R}$. Then

$$\left|\int_{r_R^{(1)}}\frac{e^{iz}}{z}\,\mathrm{d}z\right| \leqslant |r_R^{(1)}|\cdot\frac{1}{R} \leqslant C\cdot\sqrt{R}\cdot\frac{1}{R} = \frac{C}{\sqrt{r}} \to 0 \text{ as } R\to\infty$$

and

$$\left|\int_{r_R^{(2)}}\frac{e^{iz}}{z}\,\mathrm{d}z\right| \leqslant 2\pi R\frac{e^{-R^{1/2}}}{R} = 2\pi e^{-R^{1/2}} \to 0 \text{ as } R\to\infty.$$

The part for $r_R^{(3)}$ is the same as the first.

Going back to the original four sums, we obtain

$$\int_{-R}^{-\epsilon}\frac{e^{ix}}{x}\,\mathrm{d}x + \int_{\epsilon}^{R}\frac{e^{ix}}{x}\,\mathrm{d}x \to \pi i \text{ as } \epsilon\to 0, R\to\infty.$$

Taking imaginary part,

$$\int_{-R}^{-\epsilon}\frac{\sin x}{x}\,\mathrm{d}x + \int_{\epsilon}^{R}\frac{\sin x}{x}\,\mathrm{d}x \to \pi,$$

so

$$\int_0^\infty\frac{\sin x}{x}\,\mathrm{d}x = \frac{\pi}{2}$$

in the improper Riemann sense.

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With the method of residues, we can evaluate integrals:

(1) $\int_0^{2\pi} R(\cos\theta,\sin\theta)\,\mathrm{d}\theta$. Use $z = e^{i\theta}$, $\cos\theta = (z+1/z)/2$, and $\sin\theta = (z-1/z)/(2\pi)$. Then the integral becomes

$$\int_{\partial\mathbb{D}}\frac{R((z+z^{-1})/2,(z-z^{-1})/2)}{iz}\,\mathrm{d}z$$

(2) P, Q polynomials with $\deg Q \geqslant \deg P + 2$ and $Q(x) \neq 0$ on the real axis. Then

$$\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}\,\mathrm{d}x = 2\pi i\sum_{z_0\in Q^{-1}(\{0\})\cap\{\Im\mathfrak{m} z>0\}}\mathrm{Res}(P/Q, z_0).$$

(3) P, Q polynomials with $\deg Q \geqslant \deg P + 2$ and $Q(x) \neq 0$ on the real axis. Then

$$\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}e^{ix}\,\mathrm{d}x = 2\pi i\sum_{z_0\in Q^{-1}(\{0\})\cap\{\Im\mathfrak{m} z>0\}}\mathrm{Res}(e^{(iz)}P(z)/Q(z), z_0).$$

(4) If $\deg Q \geqslant \deg P + 1$ and $Q(x) \neq 0$ on the real axis then

$$\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}e^{ix}\,\mathrm{d}x = \ldots$$

as an improper Riemann integral.

(5) $\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}\sin x\,\mathrm{d}x$ where $\deg(Q) \geqslant \deg P + 1$ and Q has a simple zero at 0. Use the upper half disk minus an upper half ϵ-disk as used in the example of $\sin x/x$.

(6) $\int_0^{\infty} x^\alpha\frac{P(x)}{Q(x)}\,\mathrm{d}x$ where $\deg Q \geqslant \deg P + 2$ and $\alpha\in(0,1)$.

Consequences of the Residue Theorem

Theorem 4.7.4: Argument Principle

Assume that f is meromorphic (analytic except at poles) in Ω with zeros a_j and poles b_j repeated according to multiplicity. Then for every cycle $\gamma \in C_0$ such that $\gamma \sim 0 \pmod{\Omega}$ and γ does not pass through any poles or zeros, we have

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,\mathrm{d}z = \sum_j n(\gamma, a_j) = \sum_j n(\gamma, b_j).$$

More generally,

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)} g(z)\,\mathrm{d}z = \sum_j n(\gamma, a_j) g(a_j) - \sum_j n(\gamma, b_j) g(b_j).$$

Proof. If f has a zero of order n at a, then $f(z) = (z-a)^n g(z)$ where g is analytic and $g(a) =\neq 0$. Then

$$f'(z) = n(z-a)^{n-1} g(z) + (z-a)^n g(z)$$

so

$$\frac{f'(z)}{f(z)} = \frac{n}{z-a} + \frac{g'(z)}{g(z)}$$

where the quotient is analytic, so the residue only comes from $n/(z-a)$. Hence

$$\operatorname{Res}(f'/f, a) = m.$$

If f has a pole at a of order m, then by the same argument we have $\operatorname{Res}(f'/f, a) = -m$. Combing the cases and we obtain the claim. □

Theorem 4.7.5: Rouche's Theorem

Assume $\gamma \sim 0 \pmod{\Omega}$ and assume that $n(\gamma, z)$ is either 0 or 1 for $z \in \Omega$. Assume $|f(z) - g(z)| \leqslant |f(z)|$ for $z \in \{\gamma\}$. Then f and g have the same number of zeros enclosed by γ.

In particular, if g looks crazy but f is not, then his provides a powerful method of counting zeros.

Beginning of April 4, 2022

Proof. We know

$$|g(z)/f(z) - 1| < 1.$$

Let Γ be $F \circ \gamma$, where $F = g/f$. Then by the argument principle,

$$\frac{1}{2\pi i}\int_\gamma \frac{F'}{F} = \sum_{\text{zeros of } F} n(\gamma, a_j) - \sum_{\text{poles of } F} n(\gamma, a_j).$$

The first corresponds to the zeros of g and the second to that of f. (We can assume the zeros of F are not zeros of f for the zeros of g, for even if that happens our claim holds.) □

Example 4.7.6: Fundamental Theorem of Algebra. Consider $p(z) := z^n + a_{n-1}z^{n-1} + \ldots + a_0$. We suppose $p(z)$ and z^n are in a disk D_r where R is sufficiently large. Since for $R > 1$,

$$|p(z)/z^n - 1| = \left|\frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \ldots + \frac{a_n}{z^n}\right| \leqslant \frac{|a_{n-1}| + \ldots + |a_n|}{|z|},$$

if we set $R > \max\{1, |a_{n-1}| + \ldots + |a_n|\}$ then $|p(z)/z^n| < 1$ in ∂D_r. By Rouché, $p(z)$ and z^n have the same number of zeros in D_R, so it has n zeros in the closure.

Theorem 4.7.7: Enhanced Version of Rouché

Assume $\gamma \sim 0 \pmod{\Omega}$ and assume that $n(\gamma, z)$ is either 0 or 1. Assume f, g are meromorphic in Ω with no zero and pole on $\{\gamma\}$. If

$$|f(z) - g(z)| < |f(z)| + |g(z)|,$$

then (the number of zeros of f inside γ) minus (the number of poles of f inside γ) equals (the number of zeros of g inside γ) minus (the number of poles of g inside γ).

Proof. We write

$$\left|\frac{f(z)}{g(z)} - 1\right| < \left|\frac{f(z)}{g(z)}\right| + 1.$$

Then $f(z)/g(z) \notin (-\infty, 0]$. Since $\{\gamma\}$ is compact,

$$h(z) := \frac{f(z)}{g(z)} \notin (-\infty, 0]$$

in an open neighborhood Ω_0 of $\{\gamma\}$. Let $\Omega' := h(\Omega_0)$. This is open (WLOG f/g is not constant) not containing zero. Therefore log is defined. With the principle branch,

$$(\log h)' = (\log f - \log g)' = \frac{f'}{f} - \frac{g'}{g}$$

so $f'/f - g'/g$ has a primitive in Ω_0. Hence

$$0 = \frac{1}{2\pi i}\int_\gamma \frac{f'}{f} - \frac{g'}{g}.$$

The claim then follows from using $\gamma \sim 0 \pmod{\Omega}$. □

Chapter 5

More Topics

5.1 Uniform Limits of Analytic Functions

Theorem 5.1.1

Suppose f_n is analytic in $\Omega_n \subset \mathbb{C}$ and f_n converges to f defended on Ω uniformly in compact subsets of Ω. Then f is analytic in Ω.

Moreover, f_n' converges to f' uniformly on compact subsets of Ω as well.

This does not hold in calculus: $n^{-1}\sin(nx) \to 0$ uniformly in $[0,1]$ but the derivatives $\cos(nx)$ does not converge in any sense.

The assumptions imply that

$$\Omega \subset \liminf_{n\to\infty} \Omega_n = \bigcup_{j=1}^{\infty} \bigcap_{i=j}^{\infty} \Omega_i$$

Beginning of April 6, 2022

More formally, we say f_n defined on Ω_n converge to f on Ω **uniformly on compact sets** (subsets of Ω) if, for all $\epsilon > 0$ and for all $K \subset \Omega$ compact, there exists $n_0 \in \mathbb{N}$ such that

$$n \geqslant n_0 \implies K \subset \Omega_n \text{ and } |f_n(x) - f(x)| \leqslant \epsilon \qquad \text{for all } x \in K.$$

Alternatively we can replace K by any disk whose closure is in Ω. Typically we either have $\Omega_n = \Omega$ or $\Omega_n \nearrow \Omega$.

Proof. Let $z_0 \in \Omega$ and $r > 0$ be such that $\overline{D_r(z_0)} \subset \Omega$. Since $f_n \to f$ uniformly on $\overline{D_r(z_0)}$, we get

$$\int_\gamma f \, \mathrm{d}z = 0$$

for every γ in C_0 w.r.t. $D_r(z_0)$. □

Assertion: let $\overline{D_r(z_0)} \subset \Omega$. Then

$$f_n'(z) = \frac{1}{2\pi i} \int \frac{f_n(\zeta)}{(\zeta - z)^2} \, \mathrm{d}\zeta$$

for all $z \in D_r(z_0)$. Since $(\zeta - z)^2$ is bounded, for fixed z, $f_n(\zeta)/(\zeta - z)^2$ converges uniformly to $f(\zeta)/(\zeta - z)^2$. Passing to the limit on a compact set we obtain

$$\lim_{n\to\infty} f_n'(z) = \frac{1}{2\pi i}\int_{\partial D_r(z_0)} \frac{f_n(\zeta)}{(\zeta - z)^2}\,\mathrm{d}\zeta = f'(z)$$

where the last = is by Cauchy. Observe that this is uniform in compact subsets of $D_r(z_0)$ as well.

Corollary 5.1.2

Assume f_n is analytic in Ω_n and assume that

$$f(z) = f_1(z) + f_2(z) + \dots$$

converges uniformly on compact subsets of Ω. Then f must be analytic in Ω.

Theorem 5.1.3: Hurwitz

Assume that each f_n is analytic and nonzero in Ω and assume that $f_n \to f$ uniformly in compact subsets of Ω. Then either $f = 0$ or f has no zeros in Ω.

Beginning of April 8, 2022

Proof. Assume that f is not identically 0. Let $\overline{D_r(a)} \subset \Omega$ and assume that

$$f^{-1}(\{0\}) \cap \partial D_r(a) = \varnothing,$$

which is always possible since zeros cannot accumulate. Then there exists $c_0 > 0$ such that $|f| \geqslant c_0$ in $\partial D_r(a)$. Therefore

$$0 = \frac{1}{2\pi i}\int_{\partial D_r(a)} \frac{f_n'(z)}{f_n(z)}\,\mathrm{d}z \to \frac{1}{2\pi i}\int_{\partial D_r(a)} \frac{f'(z)}{f(z)}\,\mathrm{d}z$$

where the first = is because f_n has no zero. The RHS is the number of zeros of f in $D_r(a)$, counting multiplicity. □

5.2 Taylor Series

Recall that if f is analytic in $D_r(z_0)$ then

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \dots + \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n + f_{n+1}(z)(z - z_0)^{n+1}$$

where

$$f_{n+1}(z) = \frac{1}{2\pi i}\int_{\partial D_r(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}(\zeta - z)}\,\mathrm{d}\zeta.$$

Also,

$$|f_{n+1}(z)(z - z_0)^{n+1}| \leqslant \frac{1}{2\pi} 2\pi r \frac{\sup_{\partial D_r(z_0)}|f(z)||z - z_0|^{n+1}}{r^{n+1}(r - |z - z_0|)}$$

Theorem 5.2.1

Assume f is analytic in Ω and $z_0 \in \Omega$. Then

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

for $|z - z_0| < r$ where r is the distance between 0 and $\partial\Omega$.

If the radius r of the series is strictly greater than the distance between z_0 and $\partial\Omega$, then f can be extended to an analytic function in $\Omega \cup D_r(z_0)$.

If we have two series, $\sum_{i=0}^{\infty} a_i z^i$ and $\sum_{j=0}^{\infty} b_j z^j$ with radii of convergence r_1, r_2, then

$$fg = a_0 b_0 + (a_0 b_1 + a_1 b_0)z + (a_0 b_2 + a_1 b_1 + a_2 b_0)z^2 + \ldots,$$

where the coefficients convolve, has radius of convergence $\min(r_1, r_2)$ since fg is analytic and its n^{th} derivative is precisely the convolution of the first n terms.

How about inverse? Given f, find g such that $g(f(z)) = z$? We want $b_0, b_1, \ldots$ such that

$$b_0 + b_1(a_0 + a_1 z + a_2 z^2 + \ldots) + b_2(z_0 + a_1 z + a_2 z^2 + \ldots)^2 + \ldots = z.$$

Beginning of April 11, 2022

For a product, define

$$f(z) := a_0 + a_1(z - z_0) + z_2(z - z_0)^2 + \ldots$$

and

$$g(z) = b_0 + b_1(z - z_0) + b_2(z - z_0)^2 + \ldots$$

so that

$$f(z)g(z) = c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \ldots$$

where

$$c_n = (n!)^{-1} \frac{\mathrm{d}^n}{\mathrm{d}z^n}(f(z)g(z)) = (n!)^{-1} \sum_{j=0}^{n} \binom{n}{j} \frac{\mathrm{d}^j f(z_0)}{\mathrm{d}z^j} \frac{\mathrm{d}^{n-j} g(z_0)}{\mathrm{d}z^{n-j}}.$$

Going back to the “inverse”, we see that

$$\begin{aligned}
1 &= a_1 b_1 \\
0 &= b_1 a_2 + b_2 a_1^2 \\
0 &= b_1 a_3 + 2b_2 a_1 a_2 + b_3 a_1^3 \\
&\ldots
\end{aligned}$$

so that we know $b_1 = 1/a_1$ from the first equation, b_2 from the second, and b_3 from the third.

5.3 Laurent Series

Recall that $a_0 + a_1 z + a_2 z^2 + \ldots$ converges for $|z| < R$ where $R \in [0, \infty)$ is its radius of convergence. For aanlytic function and $R > 0$,

$$a_0 + \frac{a_1}{z} + \frac{a_2}{z_2} + \ldots$$

therefore converges for $|z| > 1/R$ if the function corresponding to these coefficients is analytic for $|z| > 1/R$.

The series

$$\dots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$$

is called a **Laurent series** and converges in some region $R_1 < |z-z_0| < R_2$ (an annulus).

Theorem 5.3.1: Laurent Development

Assume that f is analytic in a ring $A_{R_1,R_2} := \{R_1 < |z-z_0| < R_2\}$ where $0 \leqslant R_1 < R_2 \leqslant \infty$. Then we have

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

for every $z \in A_{R_1,R_2}$, with

$$a_n := \frac{1}{2\pi i}\int_{|\zeta-z_0|=r} \frac{f(\zeta)}{(\zeta-z_0)^{n+1}}\,\mathrm{d}\zeta, \qquad \text{for any/some } r \in (R_1, R_2).$$

If $R_1 = 0$ (i.e., isolated singularity at origin), then

$$\operatorname{Res}(f, z_0) = a_{-1}.$$

Proof. WLOG assume $z_0 = 0$. Let $\gamma = \gamma_1 + \gamma_2$ where $\gamma_1 = -\partial D_{r_1}$ and $\gamma_2 = \partial D_{r_2}$. We get

$$f(z) = -\frac{1}{2\pi i}\int_{\partial D_{r_1}} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta + \frac{1}{2\pi i}\int_{\partial D_{r_2}} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta$$

for all z in between. We call the first term f_1 and the second f_2.

Then more generally, for $z \in \Omega \backslash \{\gamma\}$,

$$n(\gamma, z) f(z) = \frac{1}{2\pi i}\int_{\gamma} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta.$$

Consider first

$$f_2(z) := \frac{1}{2\pi i}\int_{\partial D_{r_2}} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta$$

which is analytic for $|z| < R_2$. Moreover,

$$f_2^{(j)}(z) = \frac{j!}{2\pi i}\int_{\partial D_{r_2}} \frac{f(\zeta)}{(\zeta - z)^{j+1}}\,\mathrm{d}\zeta$$

so

$$\frac{f_2^{(j)}(z_0)}{j!} = \frac{1}{2\pi i}\int_{\partial D_{r_2}} \frac{f(\zeta)}{(\zeta - z_0)^{j+1}}\,\mathrm{d}\zeta.$$

Therefore

$$f_2(z) = a_0 + a_1 z + a_2 z^2 + \dots$$

□

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Proof Continued. Now we analyze f_1. We rewrite it as

$$f_1(z) = \frac{1}{2\pi i}\int_{\partial D_{r_1}} \frac{f(\zeta)}{z - \zeta}\,\mathrm{d}\zeta = \frac{1}{2\pi i}\int_{\partial D_{r_1}} \frac{f(\zeta)/z}{1/\zeta/z}\,\mathrm{d}\zeta$$

which can be written as a series:

$$\frac{1}{2\pi i}\int_{\partial D_{r_1}} \frac{1}{z} f(\zeta)(1+\zeta/z) + \zeta^2/z^2 + \ldots \,\mathrm{d}\zeta$$

where the limit commutes with the integral, so it equals

$$\frac{1}{z}\frac{1}{2\pi i}\int_{\partial D_{r_1}} f(\zeta)\,\mathrm{d}\zeta + \frac{1}{z^2}\frac{1}{2\pi i}\int_{\partial D_{r_2}} \zeta f(\zeta)\,\mathrm{d}\zeta + \frac{1}{z^3}\int_{\partial D_{r_2}} \zeta^2 f(\zeta)\,\mathrm{d}\zeta + \ldots$$

□

Recall that if f has a pole at z_0, then

$$f(z) = \underbrace{\frac{a_{-n}}{(z-z_0)^n} + \ldots + \frac{a_{-1}}{(z-z_0)}}_{\text{singular part}} + \underbrace{a_0 + a_1(z-z_0) + \ldots}_{\text{regular/analytic part}}.$$

Note that if the singular part has finitely many terms then it is a removable singularity or a pole, whereas infinitely many terms correspond to an essential singularity.

Theorem 5.3.2: Mittag-Leffler Theorem

Assume that $\{b_n\}_{n=1}^{\infty}$ are different and $b_n \to \infty$. Let $P_n(\zeta)$ be polynomials without constant coefficient. Then there exists a meromorphic function f with poles at b_n with the singular part $P_n(1/(\zeta - b_n))$. Moreover, every such function can be written as

$$f(z) = \sum_n (P_n(1/(z-b_n)) - p_n(z)) + g(z)$$

where p_n are polynomials and g entire.

Proof. WLOG assume $b_n \neq 0$. The function $P_n(1/z - b_n)$ is analytic in $|z| < b_n$. Then there exists $\tilde{n}_n$ such that

$$\left| P_n\left(\frac{1}{z-b_n}\right) - p_n(z) \right| < \frac{\epsilon}{2^n} \qquad \text{for } |z| < \frac{|b_n|}{z} \qquad \text{whenever } n \geqslant \tilde{n}_n.$$

Let p_n be the polynomial with index $n = \tilde{n}_n$. Then

$$f(z) = \sum_{n=1}^{\infty}\left(P_n\left(\frac{1}{z-b_n}\right) - p_n(z)\right)$$

converges uniformly on compact subsets of $\mathbb{C}\backslash\{b_1, b_2, \ldots\}$ to an analytic function. Consider

$$g := f - P_n(1/(z-b_n))$$

for a fixed n. Then g has a removable singularity at b_n, so f has a removable singularity at b_n. □

5.4 Infinite Products

Definition 5.4.1

Let $p_1, p_2, \ldots \in \mathbb{C}$. The **product**

$$p = p_1 p_2 \ldots = \prod_{i=1}^{\infty} p_n$$

converges if

(1) at most finitely many terms are zero, and

(2) there exists $k \in \mathbb{N}$ such that

$$p_k, p_k p_{k+1}, p_k p_{k+1} p_{k+2}, \cdots$$

converges to some $q \in \mathbb{C} \setminus \{0\}$.

We then say $p = p_1 p_2 \ldots p_{k-1} q$.

Note that we allow finitely many factors to be 0 but we do not want the product to converge to 0 so that (product is 0) $\Leftrightarrow$ (one of the factors is zero).

Example 5.4.2. We will see that $(1 + 1/2^2)(1 + 1/3^2)(1 + 1/4^2)\ldots$ converges. Then $0 \cdot 1 \cdot 0 \cdot 1 \cdot \ldots$ does not converge (diverges), $0 \cdot 1 \cdot 1 \cdot 1 \cdot \ldots$ converges to $0 \cdot 1 = 0$, and $0 \cdot 0 \cdot 0 \cdot \ldots$ diverges. Finally, $1 \cdot (1/2) \cdot (1/3) \cdot \ldots$ diverges since the product $\to 0$.

Proposition 5.4.3

$p_1 p_2 \ldots = 0$ if and only if there exists $j \in \mathbb{N}$ such that $p_j = 0$.

Beginning of April 15, 2022

Proposition 5.4.4

If $p = p_1 p_2 \ldots$, then $p_n \to 1$.

Proof. WLOG assume $p_\cap eq_0$ for all n. Let $P_n :=$ the first n products. By assumption $P_n \to p$ for some $p \in \mathbb{C} \setminus \{0\}$. Then since $p_{n+1} = P_{n+1}/P_n$, we have $p_{n+1} \to 1$. □

Theorem 5.4.5

Assume $1 + a_n \neq 0$ and $a_n \to 0$. Then the product $\prod_{n=1}^{\infty}(1 + a_n)$ converges iff $\sum_{n=1}^{\infty} \log(1 + a_n)$ converges.

Proof. (Idea: One direction follows from above. If the product converges then $1 + a_n \to 1$ and $1 + a_n$ eventually belongs to $\{\mathfrak{Re} z > 0\}$ so the principal branch of $\log$ is defined. Conversely, we want $\log(1 + a_n) \to 0$ and so principal branch is well-defined and $1 + a_n \to 1$.)

Denote

$$S_n := \sum_{k=1}^{n} \log(1+a_k) \qquad \text{and } P_n := \prod_{k=1}^{n}(1+a_k).$$

If S_n converges to some s, then $P_n \to e^s$.

Conversely, we assume $P_n \to p$, i.e., $e^{S_n} \to p \neq 0$. From this we see $S_n \to$ some s modulo $2\pi i$. That is, there exist k_n such that $S_n - 2\pi k_n i \to s$. This implies

$$\sum_{k=1}^{n} \log(1+a_k) - a\pi k_n i \to s,$$

so

$$\Big(\sum_{k=1}^{n+1} \log(1+a_k) - 2\pi k_{n+1} i\Big) - \Big(\sum_{k=1}^{n} \log(1+a_k) - 2\pi k_n i\Big) \to 0.$$

Hence

$$\log(1+a_{k+1}) - 2\pi(k_{n+1} - k_n)i \to 0.$$

Since P_n converges, we have $1 + a_k \to 1$, i.e., $a_k \to 0$. On the other hand $2\pi(k_{n_1} - k_n)i$ is always an integer multiple of $2\pi i$. Therefore we must have $k_{n+1} = k_n$ for sufficiently large n. This solves the modulo $2\pi i$ issue. □

Theorem 5.4.6

Assume $\mathfrak{Re} z_n > -1$ so that $\mathfrak{Re}(1+z_n) > 0$. Then

$$\sum_{n=1}^{\infty} \log(1+z_n) \text{ converges absolutely } \Leftrightarrow \sum_{n=1}^{\infty} z_n \text{ converges absolutely.}$$

As usual we are taking the principal branch of the $\log$. In either case $z_n \to 0$.

Proof. The proof relies on the estimate

$$\frac{1}{2}|z| \leqslant |\log(1+z)| \leqslant \frac{3}{2}|z|$$

for z close enough to 0, say $|z| \leqslant 1/2$. For such nonzero z,

$$\begin{aligned}\left|1 - \frac{\log(1+z)}{z}\right| &= \left|1 - \frac{z - z^2/2 + z^3/3 - \dots}{z}\right| = \left|\frac{z}{2} - \frac{z^2}{3} + \dots\right| \\ &\leqslant \frac{1}{2}(|z| + |z^2| + |z^3| + \dots) = \frac{|z|/2}{1-|z|} \leqslant 1/2.\end{aligned}$$

Therefore

$$\frac{1}{2} \leqslant \left|\frac{\log(1+z)}{z}\right| \leqslant \frac{3}{2} \qquad \text{for } |z| < \frac{1}{2}.$$

Now for $\Rightarrow$, assume $\sum|\log(1+z_n)| < \infty$. Then there exists n_0 such that $|z_n| \leqslant 1/2$ for $n \geqslant n_0$. Then

$$\sum_{n=1}^{\infty}|z_n| = \sum_{n=1}^{n_0-1}|z_n| + \sum_{n=n_0}^{\infty}|z_n| \leqslant \sum_{n=1}^{n_0+1}|z_n| + 2\sum_{n=n_0}|\log(1+z_n)|.$$

For $\Leftarrow$ we split up the sum again and do the same thing. □

Definition 5.4.7

$\prod_{n=1}^{\infty}(1+a_n)$ **converges absolutely** if $\sum_{n=1}^{\infty}\log(1+a_n)$ converges absolutely. By the previous theorem, we have $\sum_{n=1}^{\infty}|\log(1+a_n)|<\infty \Leftrightarrow \sum_{n=1}^{\infty}|a_n|<\infty$.

Theorem 5.4.8

Let $\{z_n\} \in \mathbb{C}$. Then

$$\prod_{n=1}^{\infty}(1+a_n) \text{ converges absolutely } \Leftrightarrow \sum_{n=1}^{\infty}|a_n|<\infty.$$

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Theorem 5.4.9

Assume that f_n are continuous functions on $A \subset \mathbb{C}$ to $D_z(0)$ and

$$\sum_{n=1}^{\infty}|f_n| \leqslant M < \infty \qquad \text{on } A$$

or

$$\sum_{m=1}^{\infty}|\log(1+f_n)| \leqslant M < \infty \qquad \text{on } A.$$

Then

$$\prod_{n=1}^{\infty}(1+|f_n|) \leqslant \begin{cases} e^M & \text{in the first case} \\ 3M/2 & \text{in the second case.} \end{cases}$$

5.5 Canonical Products

For motivation, we prove the following theorem:

Theorem 5.5.1

Assume that f is entire and has a finite number of zeros $a_1,\dots,a_N$ (nonzero), listed according to multiplicities. Assume 0 is of order m. Then there exists a nonvanishing entire g such that

$$f(z)=z^m\exp(g(z))\prod_{n=1}^{N}(1-z/a_n).$$

Proof. WLOG assume $N=0$ and $m>0$, for we can divide by $z^m\prod_{n=1}^{N}(1-z/a_n)$.

Now assume f is entire and has no zeros. We claim that we can write it as $\exp(g)$. Let g_0 be an entire function with $g_0'=f'/f$. Then

$$(e^{-g_0}f)'=-g_0'e^{-g_0}f+e^{-g_0}f'=0$$

so

$$e^{-g_0(z)}f(z) = \text{ constant } = e^{-g_0(0)}f(0).$$

Since $f(0) \neq 0$, there exists $a \in \mathbb{C}$ with $f(0) = e^a$. Then

$$f(z) = e^{g_0(z)}e^{-g_0(0)}e^z = e^{a-g_0(0)+g_0(z)}.$$

□

Theorem 5.5.2

(Weierstrass) Let $a_1, a_2, \ldots \in \mathbb{C}\backslash\{0\}$, be listed according to multiplicities be such that $\lim a_n = \infty$. Then

(1) there exists f entire with the zeros prescribed as above and 0 with multiplicity $m \in \mathbb{N}$.

(2) there exists $m_n \in \mathbb{N}$ (depending only on $a_1, a_2, \ldots$) such that for every function f as in (1), there exists g with

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} (1 - z/a_n) \exp(z/a_n + (z/a_n)^2/2 + \ldots + (z/a_n)^{m_n}/m_n).$$

Corollary 5.5.3

Every meromorphic f on $\mathbb{C}$ is a quotient of two entire functions.

In any bounded region, a meromorphic function is therefore a quotient of two analytic functions.

Proof. Consider zeros and poles, apply Weierstrass. □

Example 5.5.4. If we want the product

$$z^m \prod_{n=1}^{\infty} (1 - z/a_n)$$

to converge, then a sufficient condition is $\sum_{n=1}^{\infty} 1/|z_n| < \infty$. This is not necessary though; for example we can take $a_n = n$.

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Proof of Weierstrass. Consider $\prod_{n=1}^{\infty} (1 - z/a_n) \exp(p_n(z))$ where p_n are polynomials TBD. It converges if and only if

$$\sum_{n=1}^{\infty} (\log(1 - z/a_n) + p_n(z))$$

converges. Ignoring a finite amount of terms, for any z, eventually $1 - z/a_n \in D_{1/2}(1)$ uniformly on bounded sets, for we can choose the principal branch of the $\log$. Recall that

$$\log(1 - z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \ldots$$

which converges uniformly in $|z| < 1/2$. Let

$$p_n(z) := \frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \ldots + \frac{1}{m_n}\left(\frac{z}{a_n}\right)^{m_n}$$

where m_n depends on n. We need to estimate the error

$$r_n(z) = -\frac{1}{m+n+1}\left(\frac{z}{a_n}\right)^{m_n+1} - \frac{1}{m_n+2}\left(\frac{z}{a_n}\right)^{m_n+2} - \dots$$

Note that for $|z| \leqslant R$,

$$|r_n(z)| \leqslant \frac{1}{m_n+1}\left|\frac{z}{a_n}\right|^{m_n+1}\left(1+\left|\frac{z}{a_n}\right|+\left|\frac{z}{a_n}\right|^2+\dots\right) \leqslant \frac{1}{m_n+1}\left(\frac{R}{|a_n|}\right)^n \frac{1}{1-R/|a_n|}.$$

We can arrange the m_n's so that

$$\sum_{n=1}^{\infty} \frac{1}{m_n+1}\left(\frac{R}{|a_n|}\right)^{m_n+1} < \infty$$

for all $R > 0$. This is possible since $m_n = n$ works:

$$\sum_{n=1}^{\infty} \frac{1}{n+1}\left(\frac{R}{|a_n|}\right)^{n+1} < \infty.$$

Also, $m_n = n^\delta$ works for any $\delta \leqslant 1$.

With such choice of m_n's, we get

$$\sum_{n=1}^{\infty} |r_n(z)| < \infty$$

uniformly on bounded sets, so the product converges absolutely and uniformly on bounded sets. □

5.6 Riemann Mapping Theorem

Theorem 5.6.1

Let Ω be simply connected with $\Omega \neq \mathbb{C}$. Then there exists a conformal isomorphism (analytic homeomorphism) $f : \Omega \to \mathbb{D}$. Moreover, for any $z_0 \in \Omega$, there exists a *unique* $f : \Omega \to \mathbb{D}$ conformal, bijective, such that $f(z_0) = 0$ and $f'(z_0) > 0$.

Corollary 5.6.2

In $\mathbb{R}^2$, all simply connected open subsets are homeomorphic to each other.

Lemma 5.6.3

Assume that $f : \mathbb{D} \to \mathbb{D}$ is an analytic automorphism. Then there exists $a \in \mathbb{D}$ and $\varphi \in \mathbb{R}$ such that $f(z) = e^{i\varphi}(z-a)/(1-\bar{a}z)$.

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Proof. Let $f(a) = 0$. Consider

$$g(z); = \frac{f(z)}{(z-a)/(1-\bar{a}z)}$$

which is analytic with a being a removable singularity. Since g have no zeros and $|g(z)| \to 1$ as $|z| \to 1$. Applying maximum and minimum principle we see $|g(z)| = 1$ for all $z \in \mathbb{D}$. We need: for all ϵ there exists δ such that $|f(z)| \geqslant 1-\epsilon$ whenever $|z| \geqslant 1-\delta$. If this is false, then there exist a sequence $\{z_n\}$ with $|z_n| \to 1$ such that $|f(z)| <$

$1-\epsilon$. Passing to a subsequence, since the disk with radius $1-\epsilon$ is compact, there exists a convergent subsequence. Applying this to the inverse implies that z_{n_k} converges inside the disk with radius $1-\epsilon$, contradiction. □

5.7 Normal Families

We denote the set of holomorphic functions in $\Omega \subset \mathbb{C}$ (open, connected) by $\mathrm{Hol}(\Omega)$. It is possible to define a metric on $\mathrm{Hol}(\Omega)$ so that the corresponding topology is uniform convergence on compact sets.
Let $K_n \uparrow \Omega$ be an increasing sequence of compact sets. Define

$$d(f,g) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\sup_{K_n}|f-g|}{1+\sup_{K_n}|f-g|}.$$

Then $(\mathrm{Hol}(\Omega), d)$ is a metric space (and an algebra).

Definition 5.7.1

A set $\Phi \subset \mathrm{Hol}(\Omega)$ is **normal** or **relatively compact** if $\overline{\Phi}$ is compact.

Definition 5.7.2

A set $\Phi \subset \mathrm{Hol}(\Omega)$ is **uniformly bounded on compact sets** in Ω if for all $K \subset \Omega$ compact, there exists $B(K) \geqslant 0$ such that $|f(z)| \leqslant B(K)$ for all $z \in K$ and $f \in \Phi$.

Definition 5.7.3

A set $\Phi \subset \mathrm{Hol}(\Omega)$ is **equicontinuous** in a set $A \subset \Omega$ if for all $\epsilon > 0$, there exists $\delta > 0$ such that the $\epsilon - \delta$ condition holds for all $z_1, z_2 \in A$ and all $f \in \Phi$.

Theorem 5.7.4: Arzelá-Ascoli Theorem

If a family is equicontinuous on a compact set then it is relatively compact.

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Theorem 5.7.5: Normal Families

Assume $\Phi \subset \mathrm{Hol}(\Omega)$ and assume Φ is uniformly bounded on compact subsets of Φ. Then Φ is equicontinuous on every compact subset of Ω and and is also relatively compact. That is, local boundedness implies relative compactness.

Proof. Let $K \subset \Omega$ be compact and let $\delta := \mathrm{dist}(K, \partial\Omega)$. Then for all $z \in K$,

$$f'(z) = \frac{1}{2\pi i} \int_{\partial D_\delta(z)} \frac{f(\zeta)}{(\zeta - z)^2}\, \mathrm{d}\zeta$$

so

$$|f'(z)| \leqslant \frac{1}{2\pi i} 2\pi \frac{\delta}{2} \sup_{D_{\delta/2}K} |f|$$

(We need $D_{\delta/2}K$ because the derivative on the boundary of K involves points further away.)

Now for relative compactness. Let $K_1 \subset K_2 \subset \dots$ be compact with $\bigcup K_n = \Omega$, and let $f_1, f_2, \dots \in \mathrm{Hol}(\Omega)$. By Arzelá-Ascoli there exists a subsequence $f_{11}, f_{12}, \dots$ converging uniformly on K_1. Again there exists a subsequence $f_{21}, f_{22}, \dots$ of the previous subsequence converging uniformly on K_2. So forth and so on. Now consider $f_{11}, f_{22}, f_{33}, \dots$ which converges on every K_n. We are done since every compact set is contained in some K_n. $\square$

We'll work with injective mappings. Let $\Omega \subset \mathbb{C}$ be a region. If $f \in \mathrm{Hol}(\Omega)$ is $1-1$, then $f'(z) \neq 0$ for all $z \in \Omega$ and $f : \Omega \to f(\Omega)$ is an analytic isomorphism. In particular, it is open.

Proposition 5.7.6

If Ω is open, connected, f_n is $1-1$ and analytic in Ω, and $f_n \to f$ in $\mathrm{Hol}(\Omega)$, then f is $1-1$ or constant.

Proof. Assume f is not constant. Let $z_0 \in \Omega$ be arbitrary and consider

$$g_n(z) := f_n(z) - f_n(z_0).$$

Then g_0 has no zero in $\Omega \backslash \{z_0\}$. Since f is not constant, the function

$$f(z) - f(z_0)$$

has no zero in $\Omega \backslash \{z_0\}$ (Hurwitz). Since z_0 is arbitrary, f is $1-1$. $\square$

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Proof of the Riemann Mapping Theorem

Proof. **Step 1**. There exists an analytic isomorphism of Ω with an open subset of $\mathbb{D}$.
Here we require $\Omega \neq \mathbb{C}$. Then there exists $\alpha \in \Omega^c$. Since Ω is simply connected, we may define $g(z) := \log(z - \alpha)$ which is analytic in Ω and $1-1$.
We also have $g(z) \neq g(z_0) = 2\pi i$ for all $z \neq z_0$. We claim that there exists $\delta > 0$ such that $g(\Omega) \cap D_\delta(g(z_0) + 2\pi i) = \varnothing$. If not, there exists $z_n \in \Omega$ such that $g(z_n) \to g(z_0) + 2\pi i$ so $g_n \to z_0$ and $g(z_0) = g(z_0) + 2\pi i$.
Then

$$\frac{1}{g(z) - g(z_0) - 2\pi i}$$

is $1-1$ and bounded above by $1/\delta$.
Step 2. By step 1, we can WLOG assume $\Omega \subset \mathbb{D}$ and that $0 \in \Omega$ by using fractional linear transformations. (For the latter, use for example $z \mapsto (z-a)/(1-\overline{a}z)$.) That is, $\Omega \subset \mathbb{D}$ is open, simply connected, and contains origin. Consider the nonempty family

$$\Phi := \{f : \Omega \to \mathbb{D}, f \text{ is 1-1 and } f(0) = 0\}.$$

We claim that there exists f such that $|f'(0)|$ is maximal.
Denote

$$\lambda := \sup_{f \in \Phi} |f'(0)| > 0.$$

and find a sequence $f_n \in \Phi$ with $|f_n'(0)| \to \lambda$. Note that f_n is normal. By the theorem on normal families, we can assume $f_n \to f$ in $\mathrm{Hol}(\Omega)$ (uniformly on compact subsets). By the maximum principle, we have $|f(z)| < 1$

for all $z \in \Omega$. Also, f is $1-1$ by the previous proposition and nonconstant since $f'(0) \neq 0$. Therefore $f \in \Phi$ and $|f'(0)| = \lambda$.

Step 3. Let f be as in the step. Then $f(\Omega) = \mathbb{D}$. Assume not, so there exists $\alpha \in \mathbb{D}\backslash f(\Omega)$. Let

$$T(z) := \frac{z-\alpha}{1-\overline{\alpha}z}.$$

Since Ω is simply connected and $0 \notin T(f(\Omega))$, there exists $h(z) = \sqrt{T(f(z))}$. Let

$$R(z) := \frac{z-h(0)}{1-\overline{h(0)}z}$$

and consider $g(z) := R(\sqrt{T(f(z))}$. Then $g(0) = 0$ so $g \in \Phi$. It remains to prove $|g'(0)| > |f'(0)|$ for a contradiction. Let $Sz := z^2$ and consider

$$\varphi(z) := T^{-1}(S(R^{-1}(z)))$$

so that $f(z) = \varphi(g(z))$. Then $\varphi(0) = 0$ since $f(0) = g(0) = 0$ and $\varphi : \mathbb{D} \to \mathbb{D}$. Also, φ is not 1-1 so $|\varphi'(0)| < 1$ by the Schwarz lemma. Now by chain rule

$$f'(0) = \varphi'(g(0))g'(0) < g'(0).$$

Contradiction. □