Beginning of Jan.10, 2022

Definition 0.0.1

A set $U \subset (X, d)$ is **nowhere dense** if $\overline{U}$ does not contain an open ball. That is, $\overline{U}$ has empty interior.

Definition 0.0.2

A topological space X is **meager** if it is a countable union of nowhere dense sets.

Remark. Any closed sets in X with empty interior is nowhere dense.

Example 0.0.3.

- $C \subset \mathbb{R}$ the middle-third Cantor set is meager; in particular, it is a closed set with empty interior and is therefore nowhere dense. However, $C \subset C$ (as a subset of itself) is *not* meager.
- $(\mathbb{Q} \times \mathbb{Q}) \cup (\mathbb{R} \times \{0\}) \subset \mathbb{R}^2$ is meager. The first set is a countable union of singletons and the second is closed with empty interior.

Theorem 0.0.4: Baire Category Theorem

Let X be a complete metric space. Then:

(1) $\{U_i\}$, a collection of open and dense sets, then $\bigcap U_i$ is also open and dense.

(2) X is not meager.

Proof. (1) Pick $V \neq \varnothing$ open, and we need to show that $V \cap (\bigcap U_i) \neq \varnothing$.

Since U_i are open and dense, for all i, $U_i \cap V \neq \varnothing$ and is open. Hence there exists $B(x_1, r_1) \subset U_1 \cap V$ where $0 < r_0 < 1$. Assume by induction that we have (x_i, r_i) with $B(x_{n-1}, r_{n-1}) \subset U_{n-1} \cap V$. We pick the next one so that

$$\overline{B(x_n, r_n)} \subset U_n \cap B(x_n, v_n) \qquad \text{and } 0 < r_n < 2^{-n}.$$

The second condition would imply that $\overline{B(x_n, r_n)} \subset B(x_{n-1}, r_{n-1})$.

Thus, for $m, n \geqslant N$, $x_n, x_m \subset B(x_N, r_N)$, which implies that $\{x_n\}$ forms a Cauchy sequence since $r_n \to 0$. By completeness, there exists a limit point $x_n \to x$. Moreover, $x \in \overline{B(x_N, r_N)} \subset U_N \cap B(x_N, r_N) \subset U_N \cap B(x_1, v_1)$. Since $B(x_1, v_1) \subset V$ and N is arbitrary, $x \in U_N \cap V$ for all N, i.e., $x \in V \cap (\bigcap U_i)$, as claimed.

(2) If X is meager then $X = \bigcup E_i$ where $\{E_i\}$ is a countable collection of nowhere dense sets. Hence $\{\overline{E_i}^c\}$ is a collection of open dense sets. The first part implies

$$A := \bigcup_{i=1}^{\infty} \overline{E_i}^c \neq \varnothing,$$

and by De Morgan's law,

$$A^c = \bigcup_{i=1}^{\infty} \overline{E_i} \supset \bigcup_{i=1}^{\infty} E_i = X,$$

contradiction.

□

Remark. This theorem works for any X homeomorphic to a complete metric space or with a compatible topology.
In some sense, this is used to measure a "sort" of topological size of spaces. According to BCT, a set $E \subset X$ is of the **first category** if it E is meager; otherwise it's called a set of **second category**. Thus by the BCT a complete metric space is a set of the second category.

Corollary 0.0.5

If (X, d) is complete and $\{F_n\}$ is a sequence of closed sets with empty interior, then $\bigcup F_n$ also has empty interior.

Proof. Let $U_n := F_n^c$. Then U_n is open and dense. Therefore $\overline{\bigcap U_n} = X$, i.e., $\bigcup F_n = \varnothing$. □

Theorem 0.0.6

Let X be a locally compact topological space and F_n a sequence of closed sets with empty interiors. Then the interior of $\bigcup F_n$ is empty.

Example 0.0.7: First/second category vs. measure.

(1) $A \subset \mathbb{R}$ which is of second category is of measure 0.

A concrete example: let $\{q_n\}$ enumerate $\mathbb{Q}$. Define

$$A := \bigcap_{m \geqslant 1} U_m \qquad \text{where } U_m := \bigcup_{n \geqslant 1} (x_n - 2^{-n-m}, x_n + 2^{-n-m})$$

so that

$$m(A) \leqslant \inf_{m \geqslant 1} \sum_{n \geqslant 1} 2^{-n-m} = \inf_{m \geqslant 1} 2^{1-m} = 0.$$

If A were meager, then $A = \bigcup A_k$ where A_k are nowhere dense. Note that

$$\mathbb{R} = A \cup A^c = (\bigcup \overline{A_k}) \cup A^c = (\bigcup \overline{A_k}) \cup (\bigcup U_m^c).$$

The first is a countable union of nowhere dense sets as mentioned before. By construction, $\mathbb{Q} \subset U_m$, so $U_m^c \subset \mathbb{Q}^c$, so U_m^c has empty interior. Thus A being meager implies $\mathbb{R}$ is a countable union of closed, nowhere dense sets, contradicting the second claim of BCT. Hence A is not meager and must be a set of second category.