Beginning of Jan.12, 2022

Example 0.0.1: First category set with full measure. Conversely, a topologically small set can have full measure. Let $k \geqslant 1$ and $\epsilon_0 := 1/k$. Define $K_0 := [0,1]$. Define

$$I_0 := [0, 1/2 - \epsilon_0/4] \qquad \text{and } I_2 := [1/2 + \epsilon_0/4, 1],$$

and define $K_1 := I_0 \cup I_2$ (so note that $m(K_1^c) = \epsilon_0/4$).
Define $K_2 := I_{00} \cup I_{01} \cup I_{20} \cup I_{22}$, this time having $I_{00} := [0, 1/4 - \epsilon/2^4]$ and so on. We iteratively define

$$K_n := \bigcup_{\alpha \in \{0,2\}^n} I_\alpha \qquad \text{so that } \lim_{n\to\infty} m(K_n^c) = \epsilon_0 < \infty.$$

Let $\mathcal{C}_k = \bigcap_{n \geqslant 1} K_n$. This is an intersection of compact sets and is therefore compact, and

$$m(C_k) = \lim_{n\to\infty} m(K_n) = \lim_{n\to\infty} (1 - m(K_n^c)) = 1 - \epsilon = 1 - \frac{1}{k}.$$

This set almost has full measure. Now we make one of measure 1.
Since no interval can be contained in $\mathcal{C}_k$ (after sufficiently many iterations, all intervals in K_n have length sufficiently small), the interior of $\mathcal{C}_k$ is empty. Therefore $\mathcal{C}_k$ is nowhere dense.
Now if we define $A = \bigcup_{k\in\mathbb{N}} \mathcal{C}_k$, we obtain a countable union of compact, nowhere dense sets, so by the previous corollary A is meager, even though $1 \geqslant m(A) \geqslant 1 - 1/k$ for all k, implying $m(A) = 1$.

Applications of the BCT

Definition 0.0.2

For X, Y topological spaces, a mapping $f : X \to Y$ is **open** if $f(U)$ is open whenever U is open.

Definition 0.0.3

Let X, Y be NLS (normed linear spaces). A linear map $L : X \to Y$ is **bounded** if there exists $C \in \mathbb{R}$ such that

$$\|Lx\| \leqslant C\|x\| \qquad \text{for all } x \in X.$$

The space $L(X, Y)$ consists of all bounded linear operators $X \to Y$ (this is indeed a vector space equipped with the operator norm $\|\cdot\|_{\text{op}}$ defined by $\|T\| := \sup_{x\in X} \|Tx\|/\|x\|$).

Proposition 0.0.4

If Y is Banach then so is $L(X, Y)$.

Proposition 0.0.5

Let $T : V \to W$ be linear. TFAE:

(1) T is bounded;

(2) T is continuous at zero; and

(3) T is continuous everywhere.

Proof. To show (1) $\Rightarrow$ (3): if T is bounded then $\|Tx\| \leqslant C\|x\|$ for some C and all $x \in X$, so that in particular given $\epsilon > 0$, for $\|y\| < \epsilon/C$, $\|T(y)\| = \|T(x) - T(y)\| < C\epsilon/C = \epsilon$.
(3) $\Rightarrow$ (2): to show T is continuous at an arbitrary x, we use the fact that T is translation invariant. Given $\epsilon > 0$, let $\delta > 0$ be the corresponding bound for continuity at 0. Then if $\|x - y\| < \delta$,

$$\|Tx - Ty\| \leqslant \|Tx - 0 + 0 - Ty\| = \|Tx\| + \|Ty\|$$

□

Theorem 0.0.6: Open Mapping Theorem

If $T \in L(X, Y)$ with X, Y Banach and T is also surjective, then T is open.

Proof. Since T is linear, it is translation and dilation invariant. Hence it suffices to show that there exists $r > 0$ such that $B(0, r) \subset T(B(0, 1))$.
Note that $X = \bigcup_{n \geqslant 1} B_n$ where $B_n := B(0, n)$. Because T is surjective, we have $Y = \bigcup_{n \geqslant 1} T(B_n)$. By the completeness of Y and by BCT, there exists some n such that $T(B_n)$ is *not* nowhere dense.
Moreover, since $y \mapsto ny$ is a homeomorphism from $T(B_1) \to T(B_n)$, we can assume that $T(B_1)$ is the somewhere dense set. That is, there exist $y_0 \in Y$ and r such that $B(y_0, r) \subset \overline{T(B_1)}$.
Note that $T(B_1)$ is convex and symmetric (since B_1 is and T is linear). Therefore $B(-y_0, r) \subset \overline{T(B_1)}$ as well. Thus, the convex combination

$$\frac{1}{2}B(y_0, r) + \frac{1}{2}B(-y_0, r) \in \overline{T(B_1)}.$$

However, the sum of these two balls is precisely $B(0, r)$, so we've shown $B(0, r) \subset \overline{T(B_1)}$.
By linearity again,

$$B(0, 2^{-n}r) \subset T(B(0, 2^{-n})) \qquad \text{for all } n. \tag{Δ}$$

Now we show that $B(0, r/2)$ is in fact contained in $T(B_1)$. To this end pick any $y \in B(0, r/2)$. By (Δ), $y \in \overline{T(B(0, 1/2))}$, so by definition of closure there exists $x_1 \in B(0, 1/2)$ with

$$\|y - Tx_1\| < 2^{-2}r.$$

Since $y - Tx_1$ is in $B(0, 2^{-2}r)$, by (Δ) again, $y - Tx_1 \in B(0, 2^{-2}r) \subset \overline{T(B(0, 2^{-2}))}$. Thus, there exists another $x_2 \in B(0, 2^{-2})$ such that

$$\|y - Tx_1 - Tx_2\| < 2^{-3}r.$$

Inductively, for each n we have

$$y - \sum_{j=1}^{n-1} Tx_i \in B(0, 2^{-n}r) \subset \overline{T(B(0, 2^{-n})}.$$

By closure, there exists $x_n \in B(0, 2^{-n})$ satisfying

$$\left\| y - \sum_{j=1}^{n} Tx_j \right\| < 2^{-n-1}r.$$

Letting $n \to \infty$, we see that y is the limit $\sum_{j=1}^{\infty} Tx_i$. Define $x := \sum_{j=1}^{\infty} x_j$. It remains to notice that $Tx = \lim_{n\to\infty} \sum_{j=1}^{n} Tx_j = y$ and that

$$\|x\| \leqslant \sum_{j=1}^{\infty} \|x_j\| < \sum_{j=1}^{\infty} 2^{-j} = 1,$$

so indeed $y \in T(B_1)$, as claimed.

□

Corollary 0.0.7

If X, Y are Banach and $T \in L(X, Y)$ is bijective, then T is a homeomorphism.

Proof. T^{-1} is also linear and for $U \in TX$, $(T^{-1})^{-1}(U) = T(U)$. The OMT finishes the proof. □

Definition 0.0.8

Intuitively, we define the graph $\Gamma(f)$ of $f : X \to Y$ by

$$\Gamma(f) := \{(x, y) \in X \times Y \mid y = f(x)\}.$$

If f is continuous, then $\Gamma(f)$ is obviously closed in $X \times Y$.

Theorem 0.0.9: Closed Graph Theorem

Let $T : X \to Y$ be linear and $\Gamma(f)$ closed. Assume X, Y are Banach. Then $T \in L(X, Y)$.

Proof. Since Γ is a closed subspace of $X \times Y$, it is also Banach. Define π_1, π_2 as the projection of Γ onto X, Y defined by

$$\pi_1(x, Tx) := x \qquad \text{and} \qquad \pi_2(x, Tx) := Tx.$$

Note that π_1 is linear, bounded since $\|\pi_1(x, Tx)\| = \|x\| \leqslant \|x\| + \|Tx\| = \|(x, Tx)\|_{X\times Y}$ (this is how we define a norm on a product space), and bijective since $\pi_1^{-1}(x) := (x, Tx)$ serves as the inverse. Therefore by the OMT, π_1 is open and therefore π_1^{-1} is continuous. Thus $\pi_2 \circ \pi_1^{-1}$ is also continuous, but this is nothing but T itself! □