



### Theorem 0.0.1: Uniform Boundedness Theorem

Let  $X, Y$  be NLS and let  $\mathcal{A} \subset L(X, Y)$  be a collection of linear maps  $X \rightarrow Y$ . Then:

- (1) If  $\sup_{T \in \mathcal{A}} \|T(x)\| < \infty$  for all  $x$  in a non-meager set  $F$ , i.e., if the operators are pointwise bounded on  $F$ , then  $\sup_{T \in \mathcal{A}} \|T\| < \infty$ .
- (2) If  $X$  is Banach and operators in  $\mathcal{A}$  are pointwise bounded, then they are uniformly bounded.

*Proof.* (1) Define

$$E_n := \{x \mid \sup_{T \in \mathcal{A}} \|T(x)\| \leq n\} = \bigcap_{T \in \mathcal{A}} \{x \mid \|T(x)\| \leq n\}.$$

By hypothesis,  $\sup_{T \in \mathcal{A}} \|T(x)\| < \infty$  on a non meager set  $F = \bigcup E_n$ . Therefore, one of the  $E_n$ 's must contain some ball, i.e., there exists  $n$  and  $x_0, r_0$  such that  $\overline{B(x_0, r_0)} \subset E_n$ . If we rewrite  $x$  as  $x - x_0 + x_0$ , then

$$\|T(x)\| \leq \underbrace{\|T(x - x_0)\|}_{\in E_n} + \underbrace{\|T(x_0)\|}_{\in E_n} \leq n + n.$$

Therefore, for all  $T \in \mathcal{A}$ ,  $\|x\| < r_0$  implies  $\|T(x)\| \leq 2n$ , so  $\|T\| \leq 2n/r_0 < \infty$ . This completes the proof.

(2) if  $X$  is Banach, then by BCT  $X$  is not meager, and we simply use (1). □

## 0.1 Nets

Nets are in some sense a generalization of sequences that work nicely even outside metric spaces, as they are defined on arbitrary topological spaces.

We say  $X$  is **first countable** if every  $x \in X$  has a countable basis neighborhood.

If  $X$  is first countable, then for  $f : X \rightarrow Y$ , the following are equivalent:

$$(f \text{ is continuous (pre-image definition)}) \iff (x_n \rightarrow x \text{ implies } f(x_n) \rightarrow f(x)).$$

However, if  $X$  is not first countable, then sequential continuity does not imply the pre-image open set condition.

Nevertheless, the following is always true:

$$(f \text{ is continuous (pre-image definition)}) \iff (x_n \rightarrow x \text{ as nets} \Rightarrow f(x_n) \rightarrow f(x) \text{ as nets}).$$

### Definition 0.1.1: Directed set

A **directed set**  $\langle x_\alpha \rangle_{\alpha \in A}$  is a set  $A$  with a binary relation  $\leq$  satisfying

- (1) (reflexivity)  $a \leq a$  for all  $a$ ,
- (2) (transitivity) if  $a \leq b$  and  $b \leq c$  then  $a \leq c$ , and
- (3) (upper bound) for all  $a, b$ , there exists  $c$  so that  $a \leq c$  and  $b \leq c$ .

**Definition 0.1.2: Net**

A **net** in a set  $X$  is a map  $A \rightarrow X$  by  $\alpha \mapsto x_\alpha$  where  $A$  is a directed set  $\langle x_\alpha \rangle_{\alpha \in A}$ .

**Example 0.1.3.**

- (1) In  $\mathbb{N}$ , we let  $\leq$  denote the usual  $\leq$ . Then this clearly makes a directed set.
- (2)  $D = \{\{n, n+1, n+2, \dots\}, n \in \mathbb{N}\}$  with relation  $\subset$  makes a directed set.
- (3) Let  $(X, \tau_x)$  be a topological space. For  $x \in X$  consider the neighborhoods at  $x$ :

$$\mathcal{D}_x = \{u \in \tau_x : x \in U\}$$

with relation  $\subset$ .

- (4) If  $(D_1, \leq)$  and  $(D_2, \leq)$  are directed sets, then  $(D_1 \times D_2, \leq_x)$  with

$$(a, b) \leq_x (c, d) \iff a \leq c, b \leq d.$$

is a directed set.

**Definition 0.1.4**

Given a topological space  $X$  and  $E \subset X$ , a net  $\langle x_\alpha \rangle$  is **eventually** in  $E$  if there exists some  $\alpha_0 \in A$  such that for all  $\alpha$  with  $\alpha_0 \leq \alpha$ ,  $x_\alpha \in E$ . (Compare this with the definition of convergence in metric spaces.)

We say a net  $\langle x_\alpha \rangle$  is **frequently** in  $E$   $x_\alpha \in E$  i.o. (infinitely often, that is, for all  $\alpha \in A$ , there exists  $\beta$  with  $\alpha \leq \beta$  such that  $x_\beta \in E$ ).

**Definition 0.1.5: Limit points & cluster points**

A point  $x \in X$  is a **limit point** for  $\langle x_\alpha \rangle$  if for every neighborhood  $U$  of  $x$ ,  $\langle x_\alpha \rangle$  is eventually in  $U$ . A point  $x \in X$  is a **cluster point** for  $\langle x_\alpha \rangle$  if every neighborhood  $U$  of  $x$ ,  $\langle x_\alpha \rangle$  is frequently in  $U$ .

**Proposition 0.1.6**

Given a topological space  $X$ ,  $E \subset X$ , and  $x \in X$ , then  $x$  is an **accumulation point** of  $E$  if and only if there is a net in  $E - \{x\}$  converging to  $x$ . In addition,  $x \in \overline{E}$  if and only if there is a net in  $E$  converging to  $x$ .

*Proof.* Assume  $x$  is an accumulation point of  $E$ . Let  $D_x$  be the set of neighborhood of  $x$  directed by  $\supset$  (reverse inclusion). Clearly  $X \supset U$  for all  $U \in D_x$ . Also, for each  $U \in D_x$  we pick  $x_u \in (U - \{x\}) \cap E$ . By definition of nets, as  $U$ 's are decreasing, we have  $x_u \rightarrow x$  in a net.

Conversely, if  $x_\alpha \rightarrow x$  and  $x_\alpha \in E - \{x\}$ , then every neighborhood of  $x$ ,  $U - \{x\}$ , eventually contains  $\langle x_\alpha \rangle$ . In particular it contains one  $x_\alpha$ , so  $x$  is an accumulation point.

Now for the second part: if  $x_\alpha \rightarrow x$  and  $x_\alpha \in E$  then  $x \in E$ . It is either already an accumulation point (which we are done) or we can take the constant net.  $\square$