Beginning of Jan.19, 2021

**Proposition 0.0.1: Sequential Continuity**

$f : X \to Y$ is continuous at $x \in X$ if and only if for any net $\langle x_\alpha \rangle \to x$, the corresponding net $\langle f(x_\alpha)\rangle \to f(x)$.

*Proof.* If $f$ is continuous at $x$, then if $V$ is a neighborhood of $f(x)$, we have $f^{-1}(V)$ being a neighborhood of $x$. Thus if $\langle x_\alpha \rangle \to x$ then $\langle x_\alpha \rangle$ is eventually in $f^{-1}(V)$, which means that $\langle x_\alpha \rangle$ is eventually in $V$.
For the converse, if $f$ is not continuous, there exists a neighborhood $V$ of $f(x)$ such that $f^{-1}(V)$ is not a neighborhood of $X$, so $x \in \overline{f^{-1}(V^c)}$. Using the previous proposition with $\langle f(x_\alpha)\rangle$ subject to $x_\alpha \to x$, we see $f(x_\alpha) \in V^c$ so they cannot converge to $f(x)$. □

**Definition 0.0.2: Subnet**

A **subnet** of a net $\langle x_\alpha \rangle_{\alpha \in A}$ is a net $\langle y_\beta \rangle_{\beta \in B}$ with a map $\beta \mapsto \alpha_b$ such that:

(1) For all $\alpha_0 \in A$, there exists $\beta_0 \in B$ such that $\alpha_{\beta_0} \geqslant \alpha_0$ whenever $\beta > \beta_0$ (goes to infinity);

(2) $y_\beta = x_{\alpha_\beta}$ (selects a subset of the points in the first net).

Note that subnets need not to be injective.

**Proposition 0.0.3**

If $\langle x_\alpha \rangle$ is a net in $X$, then $x \in X$ is a cluster point if and only if there exists a subnet converging to $x$.

## 0.1 Compactness

We first recall the basic definition of compactness using open covers: a space $X$ is **compact** if any open cover admits a finite subcover. A subspace is **precompact** if its closure is compact.

**Definition 0.1.1: Finite Intersection Property**

A family of subsets $\{F_\alpha\}_{\alpha \in A}$ of $X$ has the **finite intersection property** if

$$\bigcap_{\alpha \in B} F_\alpha \neq \varnothing \qquad \text{for all } \textit{finite } B \subset A.$$

Let us also recall two basic facts from previous analysis:

(1) A closed subset of a compact set is compact.

*Proof.* If $F$ is compact and $\{U_\alpha\}$ covers $F$, then $\{U_\alpha\} \cup F^c$ covers $X$. The claim follows then by taking a finite subcover $U_1 \cup U_2 \cup \ldots \cup U_n \cup F^c$, since this implies $U_1 \cup U_2 \cup \ldots \cup U_n$ covers $F$. □

(2) A topological space $X$ is compact if and only if every family $\{F_\alpha\}_\alpha \in A$ of closed sets with finite intersection property satisfies $\bigcap_{\alpha \in A} F_\alpha \neq \varnothing$.

**Proposition 0.1.2: Compact Sets in Hausdorff Spaces**

(1) If $X$ is Hausdorff (i.e., we can separate any two points by open sets), $F \subset X$ compact, and $x \notin F$, then we can find disjoint open sets $U, V$ with $F \subset U$ and $x \in V$.

(2) Compact sets of a Hausdorff space are closed.

*Proof.* (1) By assumption, for each $y \in F$, there exists $V_y$ and $U_y$ such that $x \in U_y, y \in V_y$, and $U_y \cap V_y = \varnothing$. Therefore, we have an open cover $\{V_y\}_{y \in F}$ of $F$, and by compactness there exist $y_1, \ldots, y_n$ with $F \subset \{V_{y_i}\}_{i=1}^n$. If we define

$$V := \bigcup_{i=1}^{n} V_{y_i} \text{ open} \qquad \text{and} \qquad U := \bigcap_{i=1}^{n} U_{y_i} \text{ open},$$

then $V \cap U = \varnothing$; they separate $F$ and $x$, as desired.

(2) $F^c$ is closed by (a), since every $x \in F^c$ is contained in some neighborhood $\subset F^c$. □

**Example 0.1.3: Necessity of Hausdorff..** The previous proposition easily fails without assuming $X$ is Hausdorff. If $X$ is compact but not $T_2$ (Hausdorff) then it does not need to be closed.

(1) Consider $\mathbb{Z}$ with cofinite topology. This is not Hausdorff but *every* subset is compact. For example $\mathbb{Z}^+$ is compact but not closed.

(2) $\mathbb{R}$ with "two origins": take two copies of $\mathbb{R}$ with standard topology $\mathbb{R}_1, \mathbb{R}_2$ and identify every point to a pair. Limits need not to be unique – $1/n$ converges to both $0_1$ and $0_2$. Let $K$ let any compact set around one one of the copies of origins but not the other one.

**Proposition 0.1.4: Compact Hausdorff Spaces are Normal**

Compact Hausdorff spaces are normal (every two disjoint closed set have disjoint open neighborhoods).

*Proof.* If $E, F \subset X$ are closed, they are compact. Assume further that they are disjoint. For all $x \in E$, we can separate $x$ and $F$ by $x \in U_x$, $F \subset V_x$, as stated in a previous proposition. Then all $U_x$'s cover $E$ so we can subtract a subcover $\{U_{x_i}\}_{i=1}^n$. The corresponding intersections of $V_{x_i}$'s then cover $E$, and we are done. □

A few more standard facts:

(1) If $f : X \to Y$ is continuous and $F \subset X$ compact, then $f(F)$ is compact in $Y$.

(2) If $X$ is compact, then the space of continuous functions from $X \to \mathbb{F}$ is the same as the space of bounded continuous functions from $X \to \mathbb{K}$.

**Proposition 0.1.5**

If $X$ is compact and $Y$ Hausdorff, then $f : X \to Y$, a bijection, is a homeomorphism.

*Proof.* If $K \subset X$ is closed then $K$ is compact, so $f(K)$ is compact. Since $Y$ is Hausdorff, $f(K)$ is also closed. Therefore $f$ maps closed sets to closed sets. If in addition $f$ is a continuous bijection, then it is a homeomorphism, since the inverse $f^{-1}$ satisfies the "closed set condition". □