Beginning of Jan.21, 2022

Theorem 0.0.1

Let X be a topological space. Then TFAE:

(1) X is compact;

(2) All nets have cluster points; and

(3) Every net has a convergent subnet.

Proof. We've shown (2) $\Leftrightarrow$ (3).
If X is compact and $\langle x_\alpha \rangle_{\alpha \in A}$ is a net, consider the nets $E_\alpha := \{x_\beta : \beta \geqslant \alpha\}$ and note that for all $\alpha, \beta \in A$, there exists $\gamma \in A$ with $\alpha \leqslant \gamma$ and $\beta \leqslant \gamma$. Therefore $E_\alpha \subset X$ has the finite intersection property, and so do $\overline{E_\alpha}$. By compactness, $\bigcap_{\alpha \in A} \overline{E_\alpha} \neq \varnothing$. If we take an element x in this intersection and U is a neighborhood of x, then U intersects each $\overline{E_\alpha}$ which must contain some x_α. Therefore the net $\langle x_\alpha \rangle_{\alpha \in A}$ is frequent in U. This implies X is a cluster point of $\langle x_\alpha \rangle$.
Conversely, suppose X is *not* compact. Let $\{U_\beta\}_{\beta \in B}$ be an open cover with no finite subcover and let $\mathcal{A}$ be the collection of all finite subsets of B ordered by $\subset$. For $A \in \mathcal{A}$,

$$x_A \in (\bigcup_{B \in A} U_b)^c$$

forms a net $\langle x_A \rangle_{A \in \mathcal{A}}$. We want to show that $\langle x_A \rangle$ has no cluster point. If there exists one, let $\beta \in B$ with $x \in U_\beta$. If $A \geqslant \beta$ (any subset of B containing β), then $x_A \notin U_\beta$ by definition. This proves the claim. □

Locally Compact Hausdorff Spaces (LCH)

Definition 0.0.2: LCH

A space is **locally compact** if every point has a compact neighborhood. A **locally compact Hausdorff space** is what it sounds.

Proposition 0.0.3

Let X be LCH and let $U \subset X$ be open. Then given $x \in U$ there exists N compact with $x \in N \subset U$.

Proof. WLOG assume $\overline{U}$ is compact. (If not, let U_x be the compact set around x and we replace U with $U \cap U_x$). Since X is Hausdorff, we have open set V, W separating x with $\partial \overline{U}$. Then since $x \in V \subset U \backslash W$ and $\overline{V} \subset \overline{U \backslash W}$, we see $\overline{V}$ is compact and also contained in U. □

Proposition 0.0.4

If X is LCH and $K \subset U \subset X$ with K compact and U open, then there exists a precompact V open with $K \subset V \subset \overline{V} \subset U$.

Proof. By the previous proposition, for each $x \in K$ we have a compact neighborhood $N_x \subset U$. Therefore the collection $\{N_x\}_{x \in K}$ forms an open cover of K. By compactness there exist a finite subcover $N_{x_1} \cup \ldots \cup N_{x_n}$. Let $V = N_{x_1} \cup \ldots \cup N_{x_n}$ and we are done. □

Theorem 0.0.5: Urysohn's Lemma

For a LCH X and $K \subset U \subset X$ with K compact and U open, there exists a function $f \in C(X, [0,1])$ with $f\big|_K = 1$ and $f = 0$ outside a compact subset of U.

Proof. Let V be as in the previous proposition ($K \subset V \subset \overline{V} \subset U$). Then $\overline{V}$ is normal. Then we can apply Folland's 4.15 to get

$$f \in C(\overline{V}, [0,1]) \qquad \text{with } f = 1 \text{ on K and } = 0 \text{ on } \partial V.$$

We can easily extend it to X by setting $f = 0$ on $\overline{V}^c$. It remains to check continuity.
Let $E \subset [0,1]$ be a closed set. If $0 \notin E$ then $f^{-1}(E) = (f|_{\overline{V}})^{-1}(E)$; if $0 \in E$ then $f^{-1}(E) = (f|_{\overline{V}}^{-1})(E) \cup \overline{V}^c = (f|_{\overline{V}})^{-1}(E) \cup V^c$ since $\partial\overline{V} \subset (f|_{\overline{V}})^{-1}(E)$. Since $f|_{\overline{V}}$ is continuous, $(f|_{\overline{V}})^{-1}(E)$ is closed. Also, V^c is closed. Therefore the union must be closed. Therefore f satisfies the closed set condition and is therefore continuous. □

Corollary 0.0.6

Every LCH is completely regular (closed sets / points can be separated by continuous functions).

Theorem 0.0.7: Tietze Extension Theorem

For a LCH X and $K \subset X$ compact, if $f \in C(K)$ then there exists $F \in C(X)$ such that $F\big|_K = f$. Moreover, we can make F vanish outside a compact set.