# Contents

# Chapter 1

# Measure Theory

## 1.1 Sigma Algebras

The scope of Measure Theory is to develop a consistent theory of measurement, that generalizes length, area and volume. We thus want to measure subset. The naive way of defining such a theory would be to find a function $\mu : \mathcal{P}(X) \to [0, \infty]$ that satisfies:

1) (Additivity) $\mu(E \cup F) = \mu(E) + \mu(F)$ with $E \cap F = \varnothing$;

2) (Countably Additivity) $\mu(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i)$ with $\forall i, j : E_i \cap E_j = \varnothing$;

3) $\mu(I^n) = 1$;

4) $\mu(E) = \mu(F)$ whenever $F$ is a rotation or translation of $E$.

Unfortunately the following result:

**Theorem 1.1.1: Vitali**

There is no such function.

obliges us to weaken our axioms. The only reasonable and natural thing to do is to restrict ourselves to a smaller family of subsets of $\mathcal{P}(X)$. These sets will be the measurable sets.

**Definition 1.1.2**

A collection of subsets $\varnothing \neq \mathcal{A} \subset \mathcal{P}(X)$ is an algebra if $A$ is closed under unions and complement. This collection is a $\sigma$-algebra if it is closed under countably union and complement.

**Example.** The following are trivially $\sigma$-algebras: $\mathcal{A} := \mathcal{P}(X)$ and $\mathcal{A} := \{\varnothing, X\}$. Also for $X$ a uncountable set the following family: $\mathcal{A} := \{E \subset X | E \text{ or } E^C \text{are countable}\}$ is a $\sigma$-algebra.

Observe that if $\{A_i\}_{i \in I}$ is a family of $\sigma$-algebras over the same space $X$, then $\mathcal{A} := \cap_{i \in I} A_i$ is also a $\sigma$-algebra.
If $\mathcal{B} \subset \mathcal{P}(X)$ we define $\mathcal{M}(\mathcal{B})$ to be the intersections of all $\sigma$-algebras containing $\mathcal{B}$, i.e. $\mathcal{B} \subset \mathcal{A}' \implies \mathcal{M}(\mathcal{B}) \subset \mathcal{A}'$, otherwise we have a contradiction.
We thus say that $\mathcal{M}(\mathcal{B})$ is the $\sigma$-algebra generated by $\mathcal{B}$. Note that if $\mathcal{B} \subset \mathcal{C}$ then $\mathcal{M}(\mathcal{B}) \subset \mathcal{M}(\mathcal{C})$.[1]

[1] Since any $\sigma$-algebra containing $\mathcal{C}$ will contain all the countable unions and intersection of elements of $\mathcal{B}$.

**Definition 1.1.3**

Let $X$ be a metric space, let $\mathcal{D} \subset \mathcal{P}(X)$ the space of open sets, then $\mathcal{B}_X := \mathcal{M}(\mathcal{D})$ is the family of Borel sets.

Obviously we have that $\mathcal{M}(\mathcal{C}_X) = \mathcal{B}_X$ for $\mathcal{C}_X$ the set of closed sets. So $\mathcal{B}_X$ contains open, closed, countable union of closed sets, countable intersection of open sets ...

**Proposition 1.1.4**

$\mathcal{B}_{\mathbb{R}}$ is generated by any of the following:

$$\mathcal{E}_1 := \{(a,b)\}, \mathcal{E}_2 := \{[a,b]\}, \mathcal{E}_3 := \{(a,b]\}, \mathcal{E}_4 := \{[a,b)\}$$

$$\mathcal{E}_5 := \{(a,\infty)\}, \mathcal{E}_6 := \{(-\infty,b)\}, \mathcal{E}_7 := \{[-\infty,b)\}, \mathcal{E}_8 := \{[a,\infty)\}$$

*Proof.* We want to prove $\mathcal{M}(\mathcal{E}_i) = \mathcal{B}_{\mathbb{R}}$. We have that $\mathcal{E}_i \subset \mathcal{B}_{\mathbb{R}}$ since they are all the open sets or complement of open sets or sets of the type: $\cap_{i=1}^{\infty}(a - 1/n, b) = [a,b)$.
Thus $\mathcal{M}(\mathcal{E}_i) \subset \mathcal{B}_{\mathbb{R}}$. Note that if $O \in \tau_{\mathbb{R}}$ open then:

$$O = \cup_{i=1}^{\infty} I_j, \quad I_j \text{ an interval}$$

since the open interval form a basis for $\tau_{\mathbb{R}}$. Thus $O \in \mathcal{M}(\mathcal{E}_1) \implies \mathcal{O}_X \subset \mathcal{M}(\mathcal{E}_1)$ and hence $\mathcal{B}_{\mathbb{R}} \subset \mathcal{M}(\mathcal{E}_1)$.
Observe that if $I = (a,b)$ then:

$$\forall i : I \in \mathcal{M}(\mathcal{E}_i) \implies \forall j : \mathcal{M}(\mathcal{E}_1) \subset \mathcal{M}(\mathcal{E}_j) \implies \mathcal{B}_{\mathbb{R}} \subset \mathcal{M}(\mathcal{E}_j)$$

then $\mathcal{M}(\mathcal{E}_j) = \mathcal{B}_{\mathbb{R}}$. □

Now if $\{X_\alpha\}_{\alpha \in A}$ is a collection of sets $X = \prod_{\alpha \in A} X_\alpha$ is the product space. We have coordinate maps $\pi_\alpha : X \to X_\alpha$, $\pi_\beta((x_\alpha)_{\alpha \in A}) = x_\beta$. Let $\mathcal{M}_\alpha$ be a $\sigma$-algebra on $X_\alpha$, then the product $\sigma$-algebra is generated by:

$$\mathcal{E} := \{\pi_\alpha^{-1}(E_\alpha) | E_\alpha \in \mathcal{M}_\alpha, \alpha \in A\}$$

and we denote it by $\bigotimes_{\alpha \in A} \mathcal{M}_\alpha = \mathcal{M}(\mathcal{E})$.

**Proposition 1.1.5**

If $A$ is a countable indexing family then $\bigotimes \mathcal{M}_\alpha$ is generated by the sets: $\mathcal{F} := \{\prod_{\alpha \in A} E_\alpha | E_\alpha \in \mathcal{M}_\alpha\}$.

*Proof.* We have: $\pi_\alpha^{-1}(E_\alpha) = \prod_{\beta \in A} E_\beta$ with $E_\beta = \begin{cases} E_\alpha, & \alpha = \beta \\ X_\beta, & \beta \neq \alpha \end{cases}$. In particular $\pi_\alpha^{-1}(E_\alpha) \in F$. Then, $\mathcal{E} \subset \mathcal{F} \implies \mathcal{M}(\mathcal{E}) \subset \mathcal{M}(\mathcal{F})$.
Take now $\prod_{\alpha \in A} E_\alpha \in F$, clearly this is not in $\mathcal{E}$. Then $\prod_{\alpha \in A} E_\alpha = \cap_{\alpha \in A} \pi_\alpha^{-1}(E_\alpha) \in \mathcal{M}(\mathcal{E})$ since it is countable. Therefore $F \subset \mathcal{M}(\mathcal{E})$ and thus $\mathcal{M}(\mathcal{F}) \subset \mathcal{M}(\mathcal{E})$. □

**Proposition 1.1.6**

If $\mathcal{M}_\alpha$ is generated by $\mathcal{E}_\alpha$ then $\bigotimes \mathcal{M}_\alpha$ is generated by:

$$\mathcal{F}_1 := \{\pi_\alpha^{-1}(E_\alpha) | E_\alpha \in \mathcal{E}_\alpha, \alpha \in A\}$$

Furthermore, if $A$ is countable it is generated by:

$$\mathcal{F} := \{\prod_{\alpha \in A} \mathcal{E}_\alpha | E_\alpha \in \mathcal{E}_\alpha, \alpha \in A\}$$

*Proof.* First observe that

$$\mathcal{F}_1 \subset \mathcal{E} \implies \mathcal{M}(\mathcal{F}_1) \subset_{\alpha \in A} \mathcal{M}_\alpha$$

For the other direction let $\mathcal{F}_\alpha := \{E \in X_\alpha | \pi_\alpha^{-1} \in \mathcal{M}(\mathcal{F}_1)\}$, we claim it is a $\sigma$-algebra.
Then we have that $\mathcal{E}_\alpha \subset \mathcal{F}_\alpha$ by definition of $\mathcal{F}_1$ we have that $\mathcal{M}_\alpha = \mathcal{M}(\mathcal{E}_\alpha) \subset \mathcal{F}_\alpha$ since it is a $\sigma$.algebra, then $\forall E \in \mathcal{M}_\alpha : \pi_\alpha^{-1} \in \mathcal{M}(\mathcal{F})$ and thus $\mathcal{E} \subset \mathcal{M}(\mathcal{F}_1)$ which means that $\mathcal{M}(\mathcal{E}) \subset \mathcal{M}(\mathcal{F}_1)$. The countable argument is as before. □

**Example.** On $\mathbb{R}^2$ we have $\mathcal{B}_\mathbb{R} \otimes \mathcal{B}_\mathbb{R}$ has sets generated by open boxes.

Note that if $X_1, \ldots, X_n$ are metric spaces so is the product with the product metric:

$$d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \max_{1 \leq i \leq n} \{d_i(x_i, y_i)\}$$

**Theorem 1.1.7**

Let $\{X_i\}_{i=1}^n$ be a family of metric spaces and consider $X := \prod_{i=1}^n X_i$ with the product metric, then $\bigotimes_{j=1}^n \mathcal{B}_{X_j} \subset \mathcal{B}_X$.
Furthermore, if the $X_i$ are separable[a] then $\bigotimes_{j=1}^n \mathcal{B}_{X_j} = \mathcal{B}_X$.

[a] Being separable means there is a dense countable subset.

*Proof.* The set $\bigotimes_{j=1}^n \mathcal{B}_{X_j}$ is generated by the sets $\pi_i^{-1}(U_i)$ for $u_i \in \tau_{X_i}$, let:

$$\mathcal{F}_1 := \{\pi_i^{-1}(U_i) | U_i \in \tau_{X_i}\} \subset \mathcal{O}_X$$

then $\bigotimes_{j=1}^n \mathcal{B}_{X_j} = \mathcal{M}(\mathcal{F}_1) \subset \mathcal{M}(\mathcal{O}_X) = \mathcal{B}_X$.
Now suppose that each $X_i$ is separable. Then we have $C_j \subset \tau_j$ the set of open balls with rational radii centred in the dense countable subset, note that this is still a countable family of open sets and thus any subset is going to be a countable union of elements in $C_j$. Thus $\mathcal{M}(\mathcal{O}_{X_j}) = \mathcal{M}(C_j)$ and hence $E = \{\prod_{j=1}^n E_j | E_j \in C_j\}$ generates $\bigotimes_{j=1}^n \mathcal{B}_j$. Also $X = \prod_{j=1}^n X_j$ and every open set in $X$ is a countable union of $\prod_{j=1}^n E_j$ for $E_j \in C_j$. Then $\mathcal{B}_X$ is generated by $E$ and thus $\mathcal{B}_X = \mathcal{M}(\mathcal{O}_X) = \bigotimes_{j=1}^n \mathcal{B}_{x_j}$. □

**Corollary 1.1.8**

We have that $\mathcal{B}_{\mathbb{R}^n} = \bigotimes_{i=1}^n \mathcal{B}_\mathbb{R}$.

## 1.2 Measures

**Definition 1.2.1**

A measure space is a triple $(X, \mathcal{M}, \mu)$ where $\mathcal{M}$ is a $\sigma$-algebra and $\mu : \mathcal{M} \to [0, \infty]$ satisfies:

- $\mu(\varnothing) = 0$;
- $\mu(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i)$ if the $E_i$ are mutually disjoint;

if $\mu$ only satisfies the second property for finite sums we say it is finitely additive.
Elements of $\mathcal{M}$ are called measurable sets. If $A \in \mathcal{M}$ and $\mu(A) = 0$ we say that $A$ is a null-set. If $\mu(X) < \infty$ then $\mu$ is a finite measure.
If $X = \cup_{j=1}^{\infty} E_j$ for $E_j \in \mathcal{M}$ and $\mu(E_j) < \infty$ $\mu$ is $\sigma$-finite. If for all $E$ such that $\mu(E) = \infty$ there is $F \subset E$ measurable such that $\mu(F) < \infty$ then $\mu$ is semi-finite.
Finally $\mu$ is said complete if $\forall E$ null and $F \subset E$ then $F \in \mathcal{M}$.

**Example.** Consider $\mathbb{R}^n$ with measure $\mu$ on $\mathcal{B}_{\mathbb{R}^n}$ then $\mu$ is not finite but it is $\sigma$-finite and semi-finite.

**Proposition 1.2.2**

Let $(X, \mathcal{M}, \mu)$ be a measure space. Then:

1) if $E, F \in M$ and $E \subset F$ then $\mu(E) \subset \mu(F)$; (monotonicity)

2) if $\{F_j\}_{j=1}^{\infty} \subset \mathcal{M}$ then $\mu(\cup_{j=1}^{\infty} E_j) \leq \sum_{j=1}^{\infty} \mu(E_j)$; (subadditivity)

3) if $E_i \in \mathcal{M}$ and $E_i \subset E_{i+1}$ then:

$$\mu(\cup_{j=1}^{\infty} E_i) = \lim_{n\to\infty} \mu(E_i) \quad \text{(continuity from below)}$$

4) if $E_i \in \mathcal{M}$ and $E_{i+1} \subset E_i$ with $\mu(E_1) < \infty$ then:

$$\mu(\cap_{j=1}^{\infty} E_i) = \lim_{n\to\infty} \mu(E_i) \quad \text{(continuity from above)}$$

*Proof.* 1) $E \subset F \implies F = E \cup (E^C \cap F)$ thus $\mu(E) \leq \mu(F)$.
2) Let $F_j := E_j \setminus \cup_{i=1}^{j-1} E_i$ then: $\cup_{j=1}^{\infty} E_j = \cup_{j=1}^{\infty} F_j$ but now they are disjoint: $\mu(\cup_{j=1}^{\infty} F_j) = \sum_{j=1}^{\infty} \mu(F_j) \leq \sum_{j=1}^{\infty} \mu(E_j)$.
3) With the same family as 2):

$$\mu(\cup_{j=1}^{\infty} E_j) = \mu(\cup_{j=1}^{\infty} F_j) = \sum_{j=1}^{\infty} \mu(F_j) = \lim_{n\to\infty} \sum_{j=1}^{n} \mu(F_j)$$

now $F_j = E_j \setminus E_{j-1}$ then:

$$\sum_{j=1}^{\infty} \mu(F_j) = \lim_{n\to\infty} \sum_{j=1}^{n} \mu(E_j) - \mu(E_{j-1}) = \lim_{n\to\infty} \mu(E_n)$$

thus:

$$\mu(\cup_{j=1}^{\infty} E_j) = \lim_{n\to\infty} \mu(E_n)$$

4) Let $G_j := E_1 \setminus E_j$, then $G_j \subset G_{j+1}$ ad:

$$\mu(\cup_{j=1}^{\infty} G_j) = \lim_{n\to\infty} \mu(E_1 \setminus E_j) \implies \mu(E_1 \setminus \cap_{j=1}^{\infty} E_j) = \lim_{n\to\infty} \mu(E_1 \setminus E_j)$$

by additivity and $\mu(E_1) < \infty$ we have:

$$\mu(E_1) - \mu(\cap_{j=1}^{\infty} E_j) = \lim_{n\to\infty} \mu(E_1) - \mu(E_n) \implies \mu(\cap_{j=1}^{\infty} E_j) = \lim_{n\to\infty} \mu(E_i)$$

note that $\mu(E_1) < \infty$ is a necessary condition.

□

### 1.2.1 Complete Measures

If $(X, \mathcal{M}, \mu)$ is a measure space let $\mathfrak{N} := \{A \in \mathcal{M} | \mu(A) = 0\}$ the set of null sets. We know that $\mathfrak{N}$ is closed under countable unions and intersections. We complete our measure space by defining $(X, \overline{\mathcal{M}}, \overline{\mu})$ as: $\overline{\mathcal{M}} := \{E \cup F | E \in \mathcal{M}, F \subset N \in \mathfrak{N}\}$ and $\overline{\mu}(E \cup F) := \mu(E)$.

**Theorem 1.2.3**

Let $(X, \mathcal{M}, \mu)$ be a measure space with nulls sets $\mathfrak{N} \subset \mathcal{M}$, then add all the null sets to define $\overline{\mathcal{M}} := \{E \cup F | E \in \mathcal{M}, F \subset N, N \in \mathfrak{N}\}$ then this is a $\sigma$-algebra and there is a unique extension of $\mu$ to $\overline{\mathcal{M}}$ such that $(X, \overline{\mathcal{M}}, \overline{\mu})$ is complete.

*Proof.* We have that $\overline{\mathcal{M}}$ is obviously closed under countable unions as $\mathcal{M}$ and $\mathfrak{N}$ are. Let $E \cup F \in \overline{\mathcal{M}}$ for $E \subset N \in \mathfrak{N}$ so that $E \cup F = E \cup (F \setminus E)$ so we can always assume that $E \cap F = \varnothing$.
We can write $(E \cup F) = (E \cup N) \cap (N^C \cup F)$, then we have that:

$$(E \cup F)^C = \underbrace{(E \cup N)^C}_{} \cup \underbrace{(N \cup F^C)}_{\subset N \implies \in \mathfrak{N}}$$

so $\overline{\mathcal{M}}$ is a $\sigma$-algebra.
Define $\overline{\mu}(E \cup F) := \mu(E)$ then this is well defined since if we have $E_1 \cup N_1 = E_2 \cup N_2$ then: $E_1 \subset E_2 \cup N_2 \implies \mu(E_1) \le \mu(E_2)$ and similarly $\mu(E_2) \le \mu(E_1)$.
Now for completeness: if $\overline{\mu}(E \cup F) = 0 \implies \mu(E) = 0$ so $E \subset N$ now if $G \subset E \cup F$ then $G = \varnothing \cup G$ for $G \subset E \cup F \subset E \cup N \subset N$ then $G \in \overline{\mathcal{M}}$.
Now if $A_i := E_i \cup F_i$ with $E_i \in \overline{\mathcal{M}}$ and $F_i \in N_i$ and $\forall i, j : A_i \cap A_j = \varnothing$ then $\cup_{j=1}^{\infty} A_i = \cup_{i=1}^{\infty} E_i \cup \cup_{i=1}^{\infty} F_i$. Thus:

$$\cup_{i=1}^{\infty} F_i \subset \cup_{i=1}^{\infty} N_i \in \mathfrak{N} \implies \overline{\mu}(\cup_{i=1}^{\infty} A_i) = \mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$$

then $(X, \mathcal{M}, \overline{\mu})$ is a complete measure extending $(X, \mathcal{M}, \mu)$.
Uniqueness: let $\nu$ another complete measure extending $\mu$ then:

$$\nu(E \cup F) = \nu(E) + \nu(F) = \nu(E) = \mu(E)$$

thus $\nu\big|_{\mathcal{M}} \equiv \overline{\mu}\big|_{\mathcal{M}} \implies \nu \equiv \mu$. □

### 1.2.2 Outer Measure

We cannot find a measure defined on any subset but we can find an outer measure by approximating with covers.

**Definition 1.2.4**

An outer measure is a function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ satisfying:

1) $\mu^*(\varnothing) = 0$;

2) $\mu^*(A) \leq \mu^*(B)$ whenever $A \subset B$; (monotonic)

3) $\mu^*(\cup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$; (countable sub-additivity)

A way to obtain an outer measure is the following: given $\mathcal{E} \subset \mathcal{P}(X)$ and $\rho : \mathcal{E} \to [0, \infty]$ such that:

- $\varnothing, X \subset \mathcal{E}$;
- $\rho(\varnothing) = 0$;

define $\mu^*(A) := \inf\{\sum_{i=1}^{\infty} \rho(E_i) | A \subset \cup_{i=1}^{\infty} E_i, E_i \in \mathcal{E}\}$.

**Proposition 1.2.5**

This $\mu^*$ is an outer measure.

*Proof.* Note that for any $A \in \mathcal{P}(X)$ we have a cover of $A$ by sets in $\mathcal{E}$ since any $A \subset X \in \mathcal{E}$ thus $\mu^* : \mathcal{P}(X) \to [0, \infty]$.
As $\varnothing \in \mathcal{E} : \mu^*(\varnothing) \leq \rho(\varnothing) = 0 \implies \mu^*(\varnothing) = 0$.
Now let $A \subset B$ then any cover of $B$ is a cover of $A$ thus $B \subset \cup_{i=1}^{\infty} E_j \implies A \subset \cup_{j=1}^{\infty} E_j$, thus:

$$\mu^*(A) \leq \mu^*(B)$$

by definition $\mu^*(\cup_{j=1}^{\infty} A_j)$ has a cover $A_j \subset \cup_{k=1}^{\infty} E_j^k$ such that $\mu^*(A_j) \geq \sum_k \rho(E_j^k) - \frac{\epsilon}{2^j}$ moreover, $\cup_{j=1}^{\infty} A_j \subset \cup_{j,k} E_j^k$ is countable.
Then for all $\epsilon > 0$ we have:

$$\mu^*(\cup_{j=1}^{\infty} A_j) \leq \mu^*(\cup_{j,k}^{\infty} E_j^k) \leq \sum_{j=1}^{\infty} \mu^*(A_j) + \frac{\epsilon}{2^j} \leq \sum_{j=1}^{\infty} \mu^*(A_j) + \epsilon$$

thus for $\epsilon \to 0$ we have that:

$$\mu^*(\cup_{j=1}^{\infty} A_j) \leq \sum_{j=1}^{\infty} \mu^*(A_j)$$

thus $\mu^*$ is an outer measure. □

**Definition 1.2.6**

If $\mu^*$ is an outer measure we say that $A \in \mathcal{P}(X)$ is $\mu^*$-measurable if:

$$\forall E \subset X : \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^C)$$

Note that by sub-additivity:

$$\forall E, X \in \mathcal{P}(X) : \mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^C)$$

alternatively $A$ is $\mu^*$-measurable if:

$$\forall E \in \mathcal{P}(X) : \mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^C)$$

**Theorem 1.2.7: Carathéadory**

If $\mu^*$ is an outer measure and $\mathcal{M}$ is the set of $\mu^*$-measurable sets, then $\mathcal{M}$ is a $\sigma$-algebra and $(X, \mathcal{M}, \mu^*)$ is a complete measure.

*Proof.* We first prove that $\mathcal{M}$ is a $\sigma$-algebra. By symmetry of $\mathcal{M}$: $E \cap A \cup E^C \cap A$ we have that $\mathcal{M}$ is closed under complement. Let $A$, $B \in \mathcal{M}$ then:

$$\begin{aligned}\mu^*(E) &\geq \mu^*(E \cap A) + \mu^*(E \cap A^C) \geq \\ &\geq \underbrace{\mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^C) + \mu^*(E \cap A^C \cap B)}_{\mu^*(E \cap (A \cup B))} + \mu^*(E \cap A^C \cap B^C) \geq \\ &\geq \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^C)\end{aligned}$$

since $\mu^*(E \cap (A \cup B)^C) = \mu^*(E \cap A^C \cap B^C)$ and $E \cap (A \cup B) = (E \cap A) \cup (E \cap B)$ and by sub-additivity we have the last inequality.
Thus $\mathcal{M}$ is closed under finite unions. Now pick $\{E_j\} \subset \mathcal{M}$ and consider $\cup_{j=1}^{\infty} E_j$, make the family disjoint by $F_j := E_j \setminus \cup_{k=1}^{j-1} E_k \in \mathcal{M}$. Let $B_n := \cup_{j=1}^{n} F_j$, $B := \cup_{j=1}^{\infty} F_j$, for all $n \in \mathbb{N} : B_n \in \mathcal{M}$. Then:

$$\begin{aligned}\mu^*(E \cap B_n) &= \mu^*(E \cap B_n \cap F_n) + \mu^*(E \cap B_n \cap F_n^C) = \\ &= \mu^*(E \cap F_n) + \mu^*(E \cap B_{n-1})\end{aligned}$$

By induction we have that $\mu^*(E \cap B_n) = \sum_{k=1}^{n} \mu^*(E \cap F_k)$, thus:

$$\begin{aligned}\forall n : \mu^*(E) &= \mu^*(E \cap B_n) + \mu^*(E \cap B_n^C) \\ &= \sum_{k=1}^{n} \mu^*(E \cap F_k) + \mu^*(E \cap B_n^C) \quad B_n \subset B \implies B^C \subset B_n^C \\ &\geq \sum_{k=1}^{n} \mu^*(E \cap F_k) + \mu^*(E \cap B^C) \xrightarrow{n \to \infty} \sum_{k=1}^{\infty} \mu^*(E \cap F_k) + \mu^*(E \cap B^C) \\ &\geq \mu^*(E \cap B) + \mu^*(E \cap B^C) \quad \text{by sub-addivity}\end{aligned}$$

then $B \in \mathcal{M}$.
Also for $B = \cup_{k=1}^{\infty} F_k$ disjoint and for $E = B$ we have that $\mu^*(B) = \sum_{k=1}^{\infty} \mu^*(F_k)$. Now completeness for $(X, \mathcal{M}, \mu^*)$, let $F \subset A \in \mathcal{M}$, $\mu^*(A) = 0$. Then for $E \in \mathcal{P}(X)$ we have:

$$\mu^*(F) = \mu^*(E \cap F) + \mu^*(E \cap F^C)$$

but $E \cap F \subset A$ then: $\mu^*(E \cap F) \leq \mu^*(A) = 0$, thus:

$$\mu^*(F) = \mu^*(E \cap F^C) \leq \mu^*(E)$$

and thus $F \subset \mathcal{M}$. □

**Definition 1.2.8**

An <u>elementary family</u> is a set $\mathcal{E} \subset \mathcal{P}(X)$:

1) $\varnothing \in \mathcal{E}$;

2) $A, B \in \mathcal{E} \implies A \cap B \in \mathcal{E}$;

3) $E \in \mathcal{E}$ then $E^C$ is a finite disjoint union of elements in $\mathcal{E}$.

**Example.** The families $\{[a,b]\}$ and $\{(a,\infty)\}$ are elementary.

**Proposition 1.2.9**

If $\mathcal{E}$ elementary, then: $\mathcal{A} = \{\cup_{i=1}^n E_i | E_i \text{ disjoint}, E_i \in \mathcal{E}\}$ is an algebra.

*Proof.* If $A, B \in \mathcal{E}$ and $B^C = \cup_{j=1}^N C_j$ with $C_j \in \mathcal{E}$ disjoint then $A \setminus B = \cup_{j=1}^N (A \cap C_j)$ and $A \cup B = (A \setminus B) \cup B$ where these unions are disjoint so $A \setminus B \in \mathcal{A}$ and $A \cup B \in \mathcal{A}$.
It now follows by induction that if $A_1, \dots, A_n \in \mathcal{E}$ then $\cup_{i=1}^n A_j \in \mathcal{A}$ by looking at $\cup_{i=1}^n A_i = A_n \cup \cup_{i=1}^{n-1}(A_j \setminus A_n)$ which is a disjoint union. To see that $\mathcal{A}$ is closed under complements suppose that $A_1, \dots, A_n \in \mathcal{E}$ and $A_m^C = \cup_{j=1}^{\ell_m} B_m^j$ with $B_m^j$ disjoint members of $\mathcal{E}$, then:

$$(\cup_{m=1}^n A_m)^C = \cap_{m=1}^n (\cup_{j=1}^{\ell_m} B_m^j) = \cup\{B_1^{j_1} \cap \dots \cap B_n^{j_n} : 1 \le j_m \le \ell_m, 1 \le m \le n\}$$

□

### 1.2.3 Premeasures

**Definition 1.2.10**

Let $\mathcal{A}$ be an algebra. A function $\mu_0 : \mathcal{A} \to [0,\infty]$ is a premeasure if:

1) $\mu_0(\varnothing) = 0$;

2) if $\{A_i\}_{i=1}^\infty \subset \mathcal{A}$ disjoint such that $\cup_{i=1}^\infty A_i \in \mathcal{A}$, then: $\mu_0(\cup_{i=1}^\infty A_i) = \sum_{i=1}^\infty \mu_0(A_i)$.

**Proposition 1.2.11**

If $\mu_0$ is a premeasure, let $\mu^*$ be the associated outer measure, then:

1) $\mu^*\big|_A \equiv \mu_0, A \in \mathcal{A}$;

2) $\mathcal{A} \subset \mathcal{M}$ are the $\mu^*$-measurable sets;

it then follows it can be extended to a complete measure, thus $\mu^*$ gives a measure on $\mathcal{M}(\mathcal{A})$.

*Proof.* 1) Let $A \in \mathcal{A}$ then $\mu^*(A) \le \mu_0(A)$. Now to prove the other direction if $A \subset \cup_{i=1}^\infty A_i$ then let $B_i := A \cap (A_i \setminus \cup_{j=1}^{i-1} A_j)$, this makes the family disjoint and $B_i \in \mathcal{A}$, moreover $A = \cup_{i=1}^\infty B_i$, then $\mu_0(A) = \sum_{i=1}^\infty \mu_0(B_i) \le \sum_{j=1}^\infty \mu_0(A_j)$ taking the inf we have:

$$\mu_0(A) \le \mu^*(A) \implies \mu_0(A) = \mu^*(A)$$

ii) We have $\mathcal{A} \subset \mathcal{M}$ the $\mu^*$-measurable sets. Let $A \in \mathcal{A}$ and $E \subset X$. We want $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^C)$. Approximate $E$ by a cover: $E \subset \cup_{i=1}^\infty A_i$ and we can find one such cover such that $\mu^*(E) + \epsilon \ge \sum_{i=1}^\infty \mu_0(A_i)$. So we get:

$$\mu^*(E)+\epsilon \geq \sum_{i=1}^{\infty}\mu_0(A_i) \geq \sum_{i=1}^{\infty}\mu_0(A_i\cap A)+\mu_0(A_i\cap A^C) \geq \quad (\mu_0\text{-additivity})$$
$$\geq \mu^*(\cup_{i=1}^{\infty}A_i\cap A)+\mu^*(\cup_{i=1}^{\infty}A_i\cap A^C)\geq$$
$$\geq \mu^*(A\cap E)+\mu^*(A^C\cap E) \quad (\text{monotonicity})$$

by letting $\epsilon \mapsto 0$ we get:

$$\mu^*(A\cap E)+\mu^*(A^C\cap E)\leq \mu^*(E)$$

which yields the result by our previous argument. □

**Remark.** Note that for $\mu_0$ a premeasure and $\mu^*$ its associated outer measure then for $(X,\mathcal{M},\mu^*)$ the Carathéodory Complete Measure we have that $\mathcal{M}$ satisfies $\mathcal{M}(\mathcal{A})\subset\mathcal{M}$ by ii) and by i) $\mu^*$ extends $\mu_0$ on $\mathcal{A}$.

**Theorem 1.2.12**

Let $\mu_0:\mathcal{A}\to[0,\infty]$ be a premeasure on $X$ and $\mathcal{M}:=\mathcal{M}(\mathcal{A})$. Then $\mu^*$ is a measure on $\mathcal{M}$ extending $\mu_0$ on $\mathcal{A}$. If $\nu$ is any other outer measure that extends to $\mathcal{M}$ then: $\forall E\in\mathcal{M}:\nu(E)\leq\mu^*(E)$ with equality for all $E$ with $\mu^*(E)<\infty$. If $\mu_0$ is $\sigma$-finite the extension is unique.

*Proof.* It follows from the previous proposition that $\mu^*$ is a measure on $\mathcal{M}=\mathcal{M}(\mathcal{A})$. Let $E\in\mathcal{M}$ and $E\subset\cup_{j=1}^{\infty}A_j$ for $A_j\in\mathcal{A}$ then we have that $\nu(E)\leq\sum_{j=1}^{\infty}\nu(A_j)=\sum_{j=1}^{\infty}\mu_0(A_j)$ and also $\nu(E)\leq\inf\sum_{i=1}^{\infty}\mu_0(A_i)=\mu^*(E)$.
If $A=\cup_{j=1}^{\infty}A_j$ then $\nu(A)=\lim_{n\to\infty}\nu(\cup_{k=1}^{n}A_k)=\lim_{n\to\infty}\mu_0(\cup_{i=1}^{n}A_i)=\mu^*(A)$. If $E$ is such that $\mu^*(E)$ is finite pick an approximation: $E\subset\cup_{i=1}^{\infty}A_i:=A$, then we have:

$$\mu^*(A)\leq\mu^*(E)+\epsilon \implies \mu^*(A\setminus E)\leq\epsilon$$

then:

$$\mu^*(E)\leq\mu^*(A)=\nu(A)\leq\nu(E)+\nu(A\setminus E)\leq\nu(E)+\epsilon$$

by sending $\epsilon\to 0$ we get $\mu^*(E)\leq\nu(E)$ and so equality holds.
Now suppose $\sigma$-finite, then $X=\cup_{i=1}^{\infty}A_i$ and $\mu_0(A_i)<\infty$ and we can also assume that they are disjoint by our usual trick. Then $\forall E\in\mathcal{M}:\nu(E)=\mu^*(E)$ since:

$$\nu(E)=\sum_{i=1}^{\infty}\nu(E\cap A_i)=\sum_{i=1}^{\infty}\mu^*(E\cap A_i)=\mu^*(E)$$

□

### 1.2.4 Borel Measure on Euclidean Space

A Borel measure on $\mathbb{R}$ is a measure on $\mathbb{R}$ with $\sigma$-algebra $\mathcal{B}_{\mathbb{R}}$. If $\mu$ is Borel and finite we define: $F(X):=\mu((-\infty,x])$ (this is sometimes called the distribution function of $\mu$), then $F:\mathbb{R}\to\mathbb{R}$ is increasing (by monotonicity of $\mu$) and right continuous since for $x_n\to x$ we have that $(-\infty,x]=\cap_1^{\infty}(-\infty,x_n]$ then by continuity from above we have right continuity. Similarly we can consider such functions and define $\mu$.
Terminology: The $h$-interval (half open) are of the form $(a,b]$ and $(a,\infty)$ or $\varnothing$ for $-\infty\leq a<b<\infty$. Let $\mathcal{A}$ be the collection of disjoint unions of $h$-interval, we have that $\mathcal{A}$ is an algebra and the sigma algebra generated by $\mathcal{A}$ is $\mathcal{B}_{\mathbb{R}}$.

**Proposition 1.2.13**

Let $F : \mathbb{R} \to \mathbb{R}$ be increasing and right continuous. If $(a_j, b_j]$ for $j \leq n$ are disjoint $h$-intervals, let $\mu_0(\cup_{j=1}^n (a_j, b_j]) = \sum_{j=1}^n [F(b_j) - F(a_j)]$ with $\mu_0(\varnothing) = 0$, then $\mu_0$ is a premeasure on the algebra $\mathcal{A}$.

*Proof.* If $A \in \mathcal{A}$ then it is either $(a, b]$ or $A = \cup_{j=1}^n (a_j, b_j]$ disjoint, then we can reorder it such that: $a = a_1 < b_1 = a_2 < b_2 = \ldots < b_n = b$. Then:

$$\sum_{j=1}^n F(b_j) - F(a_j) = F(b) - F(a)$$

so it is well defined on $h$-intervals. If $A = \cup_{j=1}^n I_j = \cup_{k=1}^m J_k$ both disjoint unions then:

$$\begin{aligned}\sum_{j=1}^n \mu_0(I_j) = \sum_{j=1}^n \sum_{k=1}^m \mu_0(I_j \cap J_k) = \\ = \sum_{k=1}^m \sum_{j=1}^n \mu_0(I_j \cap J_k) = \sum_{k=1}^n \mu_0(J_k)\end{aligned}$$

We need to show that if:

$$\cup_{j=1}^\infty A \in \mathcal{A} \implies \mu_0(A) = \sum_{j=1}^\infty \mu_0(I_j)$$

If $A = \cup_{k=1}^n J_k$ then each $J_k$ is a countable union of disjoint $h$-intervals: $J_k = \cup_{j=1}^\infty I_j \cap J_k$ so we only need to show it for this case.

Assume $A = I$ a $h$-interval, $\cup_{j=1}^\infty I_j := I$, we have:

$$\forall n : \mu_0(I) = \mu_=(\cup_{j=1}^n I_j) + \mu_0(I \setminus \cup_{j=1}^n I_j) \geq \mu_0(\cup_{j=1}^n I_j) \geq \sum_{j=1}^n \mu_0(I_j)$$

so we get that $\mu_0(I) \geq \sum_{j=1}^\infty \mu_0(I_j)$. For the upper bound: assume $A = (a, b]$ with $a < b < \infty$, $F$ is right continuous so there is $\delta > 0$ such that:

$$F(a + \delta) - F(a) < \epsilon$$

and there is $\delta_j > 0$ such that: $F(b_j + \delta_j) - F(b_j) < \epsilon 2^{-j}$ for $I_j = (a_j, b_j]$. Consider $[a + \delta, b]$ and cover it by intervals of the type: $\{(a_i, b_i + \delta_i)\}$, then we have a finite subcover by compactness, moreover we can assume that the cover $\{(a_i, b_i + \delta_i)\}_{i=1}^n$ respects $b_i + \delta_i \in (a_{i+1}, b_{i+1} + \delta_{i+1})$, so each interval intersects at most 2 other intervals.

As $F$ is increasing:

$$\begin{aligned}F(b_n + \delta_n) - F(a_n) &\leq \sum_{j=1}^n F(b_j + \delta_j) - F(a_j) \leq \sum_{j=1}^n F(b_j) + \epsilon 2^{-j} - F(a_j) \\ &\leq \sum_{j=1}^n (F(b_j) - F(a_j)) + \epsilon \leq \sum_{j=1}^n \mu_0(I_j) + \epsilon\end{aligned}$$

but $\mu_0(I) = F(b) - F(a) \leq F(b_n + \delta_n) - F(a_1) + \epsilon$, hence:

$$\begin{aligned}\mu_0(I) = F(b) - F(a) &\leq F(b_n + \delta_n) - F(a_1) + \epsilon \\ &\leq \sum_{j=1}^n \mu_0(I_j) + 2\epsilon \\ &\leq \sum_{j=1}^\infty \mu_0(I_j) + 2\epsilon\end{aligned}$$

by sending $\epsilon \to 0$ we have the equality. Then if $F$ is increasing and right continuous we have that $\mu_0$ is a premeasure on $\mathcal{A}$ which yields $\mu^*$ an outer measure and hence a measure $\mu$ on $\mathcal{B}_{\mathbb{R}}$ extending $\mu_0$ so such that $\mu((a, b]) =$

$F(b)-F(a)$. For $I = (-\infty, b]$ and $b < \infty$ (or $(a, +\infty]$ with $-\infty < a$) we have that for any $M < \infty$ the intervals $(a_j, b_j+\delta_j)$ cover $[-M, B]$ (or $[a, M]$) hence we have: $F(b) - F(M) \leq \sum_{i=1}^{\infty} \mu_0(I_j) + 2\epsilon$ ($F(M) - F(a) \leq \sum_{i=1}^{\infty} \mu_0(I_j) + 2\epsilon$) then by letting $\epsilon \to 0$ and $M \to \infty$ we get the result. □

**Theorem 1.2.14**

If $F$ is increasing and right continuous, there exists a unique Borel measure $\mu_F$ on $\mathbb{R}$ such that $\mu_F((a,b]) = F(b) - F(a)$. If $F$ is another such function $\mu_F = \mu_G$ if and only if $F - G$ is a constant. Conversely if $\mu$ is a Borel measure, finite on bounded intervals and $F(x) := \begin{cases} \mu((0,x]), & x > 0 \\ 0, & x = 0 \\ -\mu((x,0]), & x < 0 \end{cases}$ then $F$ is increasing right continuous and $\mu_F = \mu$.

*Proof.* The function $F$ gives a pre-measure on $\mathcal{A}$ which is $\sigma$-finite. Which then extends to a unique measure on $\mathcal{B}_{\mathbb{R}}$. If $F - G \equiv c$ then pre-measures on $\mathcal{A}$ are equal which then implies that $\mu_F = \mu_G$. We have:

$$\mu_0^F((a,b]) = F(b) - F(a) = G(b) - G(a) = \mu_0^G((a,b])$$

if $F - G \not\equiv c$ then $\mu_0^F \neq \mu_0^G$ and so $\mu_F \neq \mu_G$.
If $\mu$ is a Borel measure, the monotonicity of $\mu$ implies the monotonicity of $F$ and continuity of $\mu$ from above implies right continuity of $F$ $x \geq 0$ and $x < 0$. It is then clear that $\mu = \mu_F$ on $\mathcal{A}$ and hence $\mu = \mu_F$ on $\mathcal{B}_{\mathbb{R}}$ by the uniqueness of 1.2.15.

□

Given $F$ increasing and right continuous, then there is a unique Borel measure $\mu_F$. By Carathéadory we have some complete measure which restricts to $\mu_F$ on the Borel sets, then $\mathcal{B}_{\mathbb{R}}$: $(\mathbb{R}, \overline{\mu}_F, \mathcal{M}_{\overline{\mu}_F})$ is a complete measure and is the Lebesgue-Stieltjes measure of $F$.

### Lebesgue-Stieltjes Measure

Let $F$ be an increasing right continuous function for $(\mathbb{R}, \mu, \mathcal{M}_\mu)$ its Lebesgue-Stieltjes integral. If $E \in \mathcal{M}_\mu$ we have:

$$\mu(E) := \inf\{\sum_{j=1}^{\infty} F(b_j) - F(a_j) | E \subset \cup_{j=1}^{\infty}(a_j, b_j]\}$$

**Proposition 1.2.15**

If $\mu$ is Lebesgue-Stieltjes for $F$, with $\mathcal{M}_\mu$ the $\sigma$-algebra, then:

$$\mu(E) = \inf\{\sum_{i=1}^{\infty} \mu((a_i, b_i)) | E \subset \cup_{i=1}^{\infty}(a_i, b_i)\}$$

*Proof.* Let $\nu(E) := \inf\{\sum_{i=1}^{\infty} F(b_i) - F(a_i) | E \subset \cup_{i=1}^{\infty}(a_i, b_i)\}$ for $(a_j, b_j) = \cup_{k=1}^{\infty}(c_k^k, c_{k+1}^j]$ with $a_j = c_j^1$ and $c_k^j \to b_j$ from below. Then:

$$E \subset \cup_{j=1}^{\infty}(a_j, b_j) \subset \cup_{j=1}^{\infty} \cup_{k=1}^{\infty} (c_k^k, c_{k+1}^j]$$

so:

$$\begin{aligned}\mu(E) &\leq \sum_{j,k} \mu((c_k^k, c_{k+1}^j]) \leq \sum_j \sum_k \mu((c_k^k, c_{k+1}^j]) \\ &= \sum_j \mu((a_j, b_j)) \leq \sum_j \mu((a_j, b_j]) \\ &= \sum_j F(b_j) - F(a_j) \leq \nu(E)\end{aligned}$$

by definition of $\mu$ there are $(a_j, b_j]$ such that $E \subset \cup_j (a_j, b_j]$ with:

$$\sum_j \mu((a_j, b_j]) \leq \mu(E) + \epsilon$$

by continuity there is a $\delta_j$ such that:

$$F(b_j + \delta_j) - F(a_j) \leq \epsilon 2^{-j} \implies E \subset \cup_{j=1}^{\infty} (a_j, b_j + \delta_j)$$

so:

$$\begin{aligned}\nu(E) &\leq \sum_{j=1}^{\infty} F(b_j + \delta_j) - F(b_j) \leq \sum_{j=1}^{\infty} F(b_j) - F(a_j) + \epsilon 2^{-j} \\ &\leq \sum_{j=1}^{\infty} F(b_j) - F(a_j) + \epsilon \\ &= \mu(E) + \epsilon\end{aligned}$$

by letting $\epsilon \to 0$ we get $\nu(E) \leq \mu(E)$ and thus the equality. □

**Theorem 1.2.16**

If $E \in \mathcal{M}_\mu$:

$$\mu(E) = \inf\{\mu(U) | E \subset U, U \text{ open}\} = \sup\{\mu(K) | K \subset E, K \text{ compact}\}$$

*Proof.* Let $E \subset U$ then $\mu(E) \leq \mu(U)$ which implies that $\mu(E) \leq \inf \mu(U)$ by the prior proposition there exists $U = \cup_{j=1}^{\infty} (a_j, b_j]$ with $E \subset U$ and $\mu(U) \leq \mu(E) + \epsilon$ so $\inf_{E \subset U} \mu(U) \leq \mu(E) + \epsilon$ then by letting $\epsilon \to 0$ we get that $\inf_{E \subset U} \mu(U) \leq \mu(E)$, which yields the result.

Now for the compact sets. Assume $E$ is bounded then $\overline{E}$ is compact[2]. Consider $\overline{E} \setminus E$ by the previous point there is $U$ open such that $\overline{E} \setminus E \subset U$: $\mu(U) \leq \mu(\overline{E} \setminus E) + \epsilon$. Then $K := \overline{E} \setminus U$ compact and $K \subset E$ so:

$$\begin{aligned}\mu(K) &= \mu(E) - \mu(E \cap U) = \mu(E) - (\mu(U) - \mu(U \setminus E) = \\ &= \mu(E) - \mu(U) + \mu(U \setminus E) \\ &\geq \mu(E) - \underbrace{\mu(U) + \mu(\overline{E} \setminus E)}_{\leq \epsilon} \\ &\geq \mu(E) - \epsilon\end{aligned}$$

so by taking the sup and letting $\epsilon \to 0$ we get: $\sup_{K \subset E, cpt} \mu(K) = \mu(E)$, since the other inequality is trivial.

If $E$ is unbounded let $E_j := E \cap (j, j+1)$ then by the above for all $J$ there is $K_j \subset E_j$ with:

$$\forall j \in \mathbb{Z} : \mu(K_j) \geq \mu(E_j) - \epsilon 2^{|j|}$$

[2] We are considering $\mathbb{R}_{\text{std}}$ so Heine-Borel holds.

let $H_n := \cup_{-n}^{n} K_j$, this is a compact set and $H_n \subset E$, by construction we have:

$$\mu(H_n) \geq \mu(E) - 2\epsilon$$

so $\mu(E) = \sup_{K \subset E, cpt} \mu(K)$. □

**Theorem 1.2.17**

We have that the following are equivalent:

1. $E \in \mathcal{M}_\mu$;
2. $E = V \setminus N$ for $V \in G_\delta$-set and $N$ a null set;
3. $E = H \cup N$ for $H \in F_\sigma$-set and $N$ a null set.

*Proof.* Obviously (2) and (3) imply (1) since $\mu$ is complete on $\mathcal{M}_\mu$. Suppose $E \in \mathcal{M}_\mu$ and $\mu(E) < \infty$. By the previous Theorem 1.2.19 for $j \in \mathbb{N}$ we can choose an open set $U_j$ and a compact set $K_j \subset E$ such that:

$$\mu(U_j) - 2^{-j} \leq \mu(E) \leq \mu(K_j) + 2^{-j}$$

let $B := \cap_{j=1}^{\infty} U_j$ and $H := \cup_{j=1}^{\infty} K_j$. Then $H \subset E \subset V$ and $\mu(V) = \mu(H) = \mu(E) < \infty$, so $\mu(V \setminus H) = \mu(E \setminus H) = 0$. We thus have the result for $\mu(E) < \infty$.
For $\mu(E) = \infty$ let $E_j := E \cap [j, j+1]$, in particular there exists $H_j$, $N_j$ with $E_j = H_j \cup N_j$ $H_j \in \mathcal{F}_\sigma$ and $N_j$ a null set. Then let $H := \cup_{j=1}^{\infty} H_j \in F_\sigma$ and $N := \cup_{j=1}^{\infty} N_j$ which is still a null set, then $E = H \cup N$. So we have $(1) \implies (3)$ and then we are done since by taking complements we have $(2) \iff (3)$. □

The significance of this Theorem is that all Borel sets (or more generally all sets of $\mathcal{M}_\mu$) are of reasonably simple form modulo set of measure zero.

**Corollary 1.2.18**

We have that $(\mathcal{M}_\mu, \mu)$ is the completion of $\mu$ on $\mathcal{B}_\mathbb{R}$.

*Proof.* If $E \in \mathcal{M}_\mu$ then $E = B_1 \cup N_1$ for $B_1 \in \mathcal{B}_\mathbb{R}$ and $N_1$ a null set. □

Another version of the same idea is that general measurable sets can be approximated by `simple` sets:

**Proposition 1.2.19**

If $E \in \mathcal{M}_\mu$ and $\mu(E) < \infty$ then there is a finite collection of intervals $A$ such that $\mu(E \triangle A) < \epsilon$.

*Proof.* Choose $U$, $K$ such that $K \subset E \subset U$ open and compact respectively such that: $\mu(U \setminus E) < \epsilon$ and $\mu(E \setminus K) < \epsilon$. Take a cover of $K$ by open intervals contained in $U$ and ick a finite subcover: $K \subset A := \cup_{j=1}^{n} I_j$. So: $\mu(K) \leq \mu(A) \leq \mu(U)$ implies that:

$$\mu(E) - \epsilon \leq \mu(A) \leq \mu(E) + \epsilon$$

So we get that $\mu(E \triangle A) = \mu(E \setminus A) + \mu(A \setminus E) \leq 2\epsilon$ and we are done. □

**Definition 1.2.20**

We define the Lebesgue Measure $(\mathbb{R}, \mathfrak{L}, m)$ as the measure induced by the identity function: $F(x) = x$ so $\mu_F = m$ and $\mathcal{M}_{\mu_F} = \mathfrak{L}$.

For $E \subset \mathbb{R}$ and $r, s \in \mathbb{R}$ we define $E + s := \{x + s | x \in E\}$ and $rE := \{rx | x \in E\}$. We then have:

**Theorem 1.2.21**

If $E \in \mathfrak{L}$ then $E + s, rE \in \mathfrak{L}$ for all $r, s \in \mathbb{R}$. Moreover $m(E + s) = m(E)$ and $m(rE) = |r| m(E)$.

*Proof.* We have $\mathcal{B}_{\mathbb{R}} = \cap \mathcal{A}_i$ over all $\sigma$-algebra containing the intervals. If $\mathcal{A}_i$ is a $\sigma$-algebra then $\mathcal{A}_i + s$ is a $\sigma$-algebra and also $r\mathcal{A}_i$ for $r \neq 0$ is a $\sigma$-algebra containing open intervals. Then $\mathcal{B}_{\mathbb{R}} = \cap \mathcal{A}_i + s = (\cap \mathcal{A}_i) + s = \mathcal{B}_{\mathbb{R}} + s$ and similarly we get that $\mathcal{B}_{\mathbb{R}} = r\mathcal{B}_{\mathbb{R}}$. Now for $E \in \mathcal{B}_{\mathbb{R}}$ define $m_s(E) := m(E+s)$ and $m^r(E) := m(rE)$, $m_s$ and $m^r$ agree on unions of disjoint intervals so $m_s$ agrees on $\mathcal{B}_{\mathbb{R}}$ with $m(E)$ and similarly $m^r = |r| m$.
Then for Borel null sets $m(E) = 0$ we have that $m_s(E) = m^r(E) = 0$. So if $N$ is Lebesgue null $N \subset E$ for $E$ Borel null implies that $(N + s) \subset (E + s)$ and by completion we have $N + s$ is null. Now f $E \in \mathfrak{L}$ we have that $E = B \cup N$ for $B \in \mathcal{B}_{\mathbb{R}}$ and $N$ a null set, then we have that $m(E) = m(B) = m(B + s) = m(E + s)$. □

Now we have that countable sets have null Lebesgue measure since points have null measure.
The Cantor set: $C := x \in [0,1] | x = \sum_{j=1}^{\infty} a_j 3^{-j};\ a_j = 0, 2$. We can cover $C$ with $2^j$ intervals of length $3^{-j}$ so $\mu(C) \leq (\frac{2}{3})^j \to 0$.
The set $C$ is uncountable since $f : C \to [0,1]$ defined as $f(\sum a_j 3^{-j}) := \sum(\frac{a_j}{2}) 2^{-j}$ yields all binary expressions.
We can extend $f$ to a continuous map: $F : I \to I$ such that it is increasing and surjective by adding constant function of $C$. This function is called the Cantor Function.
To recap we have that $C$ has measure zero, is uncountable, compact, nowhere dense and a perfect set[3].

## 1.3 Integration

**Definition 1.3.1**

Given $f : X \to Y$ from $(X, \mathcal{M})$, $(Y, \mathfrak{N})$ is measurable if $\forall E \in \mathfrak{N} : f^{-1}(E) \in \mathcal{M}$.

We clearly have that measurable functions are closed under composition. If $Y = \mathbb{R}^n, \mathbb{C}^n$ we assume that $\mathfrak{N}$ is the Borel $\sigma$-algebra unless stated otherwise.
We say that $f : \mathbb{R} \to \mathbb{C}$ is Lebesgue (Borel) measurable if it is $(\mathfrak{L}, \mathcal{B}_{\mathbb{C}})$ measurable ($(\mathcal{B}_{\mathbb{R}}, \mathcal{B}_{\mathbb{C}})$ measurable). For $f, g : \mathbb{R} \to \mathbb{R}$ Lebesgue the composition $f \circ g$ does not have to be Lebesgue measurable even if $g$ is $C^0(\mathbb{R})$. However if $f$ is Borel measurable then $f \circ g$ is Lebesgue measurable or Borel measurable whenever $g$ is.

**Proposition 1.3.2**

If $\mathcal{E}$ generated $\mathfrak{N}$ then $f : X \to Y$ is measurable if and only if $f^{-1}(E) \in \mathcal{M} \forall E \in \mathcal{E}$.

[3]It does not have isolated points.

**Corollary 1.3.3**

if $F : X \to Y$ is continuous between $X, Y$ metric space then $F$ is $(\mathcal{B}_X, \mathcal{B}_Y)$-measurable.

**Corollary 1.3.4**

Let $f : X \to \mathbb{R}$ and $(X, \mathcal{M})$ a measure space. Then the following are equivalent:

1. $f$ is $\mathcal{M}$-measurable;
2. $\forall a \in \mathbb{R} : f^{-1}((a, \infty)) \in \mathcal{M}$;
3. $\forall a \in \mathbb{R} : f^{-1}((-\infty, a)) \in \mathcal{M}$;
4. $\forall a \in \mathbb{R} : f^{-1}([a, \infty)) \in \mathcal{M}$;
5. $\forall a \in \mathbb{R} : f^{-1}((-\infty, a]) \in \mathcal{M}$;

For $E \in \mathcal{M}$ define $\mathcal{M}_E := E \cap \mathcal{M}$ a new $\sigma$-algebra and $F\big|_E$ is $\mathcal{M}_E$-measurable if $F\big|_E : E \to Y$ is $(\mathcal{M}_E, \mathfrak{N})$-measurable. Given $f_\alpha : X \to Y_\alpha$ measurable $(Y_\alpha, \mathfrak{N}_\alpha)$ then there is a unique $\sigma$-algebra $\mathcal{M}$ on $X$ such that $\mathcal{M}$ is the minimal $\sigma$-algebra such that $f_\alpha$ is in $(\mathcal{M}, \mathfrak{N}_\alpha)$-measurable for all $\alpha$. The $\sigma$-algebra $\mathcal{M}$ is generated by $\{f_\alpha^{-1}(E_\alpha) | E_\alpha \in \mathfrak{g}_\alpha, \alpha \in A\}$.

**Example.** The maps $\pi_\beta : \prod_{\alpha \in A} X_\alpha \to X_\alpha$ for $(X_\alpha, \mathcal{M}_\alpha)$ gives $\otimes_{\alpha \in A} \mathcal{M}_\alpha$.

**Proposition 1.3.5**

Let $(X, \mathcal{M})$, $(Y_\alpha, \mathfrak{N}_\alpha)$ and $f : X \to \prod_{\alpha \in A} Y_\alpha$. Then $f$ is $(\mathcal{M}, \otimes \mathfrak{N}_\alpha)$-measurable if and only if $\pi_\alpha \circ f$ is $(\mathcal{M}, \mathfrak{N}_\alpha)$-measurable for all $\alpha \in A$.

*Proof.* $(\Rightarrow)$ Composition of measurable function is measurable implies that $\pi_\alpha \circ f$ is measurable.
$(\Leftarrow)$ Let $\mathcal{E} := \{\pi_\alpha^{-1}(E_\alpha) | E_\alpha \in \mathfrak{N}_\alpha, \alpha \in A\}$, generates $f^{-1}(\pi_\alpha^{-1}(E_\alpha)) = (\pi_\alpha \circ f)^{-1}(E_\alpha)$ as $\pi \circ f$ measurable implies $(\pi_\alpha \circ f)^{-1}(E_\alpha) \in \mathcal{M}$ is measurable. □

**Corollary 1.3.6**

Given $f : X \to \mathbb{R}^n$ or $(\mathbb{C}^n)$ is $\mathcal{M}$-measurable if and only if $f_i : X \to \mathbb{R}$ or $(\mathbb{C})$ is $\mathcal{M}$-measurable.

Also $f : X \to \mathbb{C}$ is $\mathcal{M}$-measurable if and only if $\mathfrak{Im}(f)$, $\mathfrak{Re}(f)$ are $\mathcal{M}$-measurable.

**Proposition 1.3.7**

If $f, g : X \to \mathbb{C}$ are $\mathcal{M}$-measurable then $f + g$, $fg$ are $\mathcal{M}$-measurable.

*Proof.* Just consider composition with continuous functions which we know to be measurable: let $F(x) = (f(x), g(x)) \in \mathbb{C}^2$, then consider $\phi_1(z, w) = z + w$ and $\phi_2(z, w) = zw$ these are continuous functions hence $(\mathcal{B}_{\mathbb{C}^2}, \mathcal{B}_{\mathbb{C}})$-measurable, so $\phi_i \circ F$ is $(\mathcal{M}, \mathcal{B}_{\mathbb{C}})$-measurable and so $f + g$, $fg$ are. □

### 1.3.1 Limits

Consider the extended real line $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\} = [-\infty, \infty]$. We can put a metric by considering $p(x,y) := \left|\tan^{-1}(x) - \tan^{-1}(y)\right|$. With Borel sets ($\mathcal{B}_{\overline{\mathbb{R}}} := \{E \subset \overline{\mathbb{R}} | E \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}}\}$) generated by the intervals $\{(a,\infty]\}$ or $\{[-\infty, a)\}$. Let $f : X \to \overline{R}$ be $\mathcal{M}$-measurable if it is $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$-measurable.

**Proposition 1.3.8**

Let $f_n : X \to \overline{\mathbb{R}}$ be $\mathcal{M}$-measurable. Then $\sup_n f_n(x)$, $\inf_n f_n(x)$ and $\limsup_{n\to\infty} f_n$, $\liminf_{n\to\infty} f_n(x)$ are $\mathcal{M}$-measurable function. Also if $f_n \to f$ point wise then $f$ is $\mathcal{M}$-measurable.

*Proof.* Let $g(x) := \sup_n f_n(x)$ then $g^{-1}((a,\infty]) = \cup_{n=1}^{\infty} f_n^{-1}((a,\infty]) \in \mathcal{M}$. Similarly: $h(x) := \inf_n f_n(x)$ then $h^{-1}([-\infty, a)) = \cup_{n=1}^{\infty} f_n^{-1}([-\infty, a))$. So $\sup f_n$, $\inf f_n$ are measurable.
So $g_k(x) := \sup_{n\geq k} f_n$ is a measurable function, then we have that

$$\limsup_{n\to\infty} f_n(x) = \lim_{k\to\infty} g_k(x) = \inf_k g_k(x)$$

is a measurable function. Similarly we can do $\liminf f_n(x)$.
If $f_n \to f$ point wise then $f(x) = \liminf_{n\to\infty} f_n(x) = \limsup_{n\to\infty} f_n(x)$ so $f$ is measurable. □

**Corollary 1.3.9**

If $f, g : X \to \overline{\mathbb{R}}$ are measurable so are $\max(f,g)$ and $\min(f,g)$.

For complex valued functions we have:

**Corollary 1.3.10**

If $f_n : X \to \mathbb{C}$ are measurable and $f_n \to f$ point wise then $f$ is measurable.

Given $f : X \to \overline{\mathbb{R}}$, let $f^+ := \max(f, 0)$ and $f^- := \max(-f, 0)$, then $f = f^+ - f^-$ and moreover we have the following result:

**Lemma 1.3.11**

The function $f$ is measurable if and only if $f^+$, $f^-$ are.

Similarly we can do the polar decomposition, so given $f : X \to \mathbb{C}$ then $f(x) = \operatorname{sgn}(f(x))\,|f(x)|$, where

$$\operatorname{sgn}(z) := \begin{cases} z/|z|, & z \neq 0 \\ 0, & \text{otherwise} \end{cases}.$$

Note that $|f|$ is measurable whenever $f$ is, since it is the composition of a continuous and measurable function.

### 1.3.2 Step Functions

If $E \subset X$, let $\chi_E : X \to \mathbb{R}$ be the function defined as:

$$\chi_E(x) := \begin{cases} 1, & x \in E \\ 0, & \text{o/w} \end{cases}$$

note that $\chi_E$is measurable if and only if $E$ is measurable. A function is said to be simple if it can be written as $f = \sum_{i=1}^n c_i\chi_{E_i}$ for $c_i \in \mathbb{C}$ and $E_i \in \mathcal{M}$. Note that simple functions are measurable.

**Lemma 1.3.12**

A map $f$ is simple if and only if $f$ is measurable and the range is a finite set of $\mathbb{C}$.

If the range of $f$ is finite and $f$ is measurable then $f^{-1}(\operatorname{im} f) = \cup_{i=1}^n E_i$, clearly if $f$ is simple then its image is a finite set of $\mathbb{C}$.

**Theorem 1.3.13**

Let $(X, \mathcal{M})$ be a measurable space:

a) if $f : X \to [0,\infty]$ is measurable the there exists $\phi_n$ simple non negative such that $\forall n : \phi_n(x) \leq f(x)$ and $\phi_n \xrightarrow{n\to\infty} f$ uniformly on any set on which $f$ is bounded on;

b) if $f : X \to \mathbb{C}$ is measurable we have the same sequence except that it converges uniformly whenever $|f|$ is bounded.

*Proof.* (a) Consider for $0 \leq n \leq \infty$ and $0 \leq k \leq 2^n-1$ the following sets: $E_n^k := f^{-1}(k2^{-n}, (k+1)2^{-n})$ and $F_n := f^{-1}((2^n, \infty])$. Then define:

$$\phi_n := \sum_{k=0}^{2^{2n}-1} k2^{-n}\chi_{E_n^k} + 2^n\chi_{F_n}$$

this function satisfies $\phi_n \leq f$ and also $\phi_n \leq \phi_{n+1}$, moreover on the set in which $f \leq 2^n$ the difference $0 \leq f - \phi_n \leq 2^{-n}$ so the result stated follows.

(b) Just apply part (a) to the positive and negative part of $im(f)$ and $re(f)$ and then let $\phi_n := \phi_n^+ - \phi_n^-$. □

**Proposition 1.3.14**

If $\mu$ is a complete measure, we have:

1. if $f$ is measurable and $g = f$ $\mu$-a.e.[a] then $f$ is measurable;
2. if $f_n$ are measurable and $f_n(x) \to f(x)$ $\mu$-a.e. then $f$ is measurable.

[a] This is point wise except on a set of measure zero.

*Proof.* (1) Let $f = g$ except on $E$ with $\mu(E) = 0$, then for any set $F \in \mathcal{M}$ we have $g^{-1}(F) = g^{-1}(F) \cap E \cup g^{-1}(F) \cap E^C$ and $g^{-1}(F) \cap E \subset E$ so it is a null set since it is a complete measure and $g^{-1}(F) \cap E^C = f^{-1}(F) \cap E^C$ is measurable.
(2) If $f_n \to f$ $\mu$-a.e. then $f_n(x) \to f(x)$ except on a set $E$ of measure zero. Let $g_n := f_n\chi_{E^C}$ a measurable function, then $g_n(x) \to f(x)\chi_{E^C}(x)$ implies that $f\chi_{E^C}$ is a measurable function and $f = f\chi_{E^C}$ $\mu$-a.e.. □

**Proposition 1.3.15**

Let $(X, \mathcal{M}, \mu)$ be a measure space and $(X, \overline{\mathcal{M}}, \overline{\mu})$ its completion. If $f$ is $\overline{\mu}$ measurable then there is $g$ $\mathcal{M}$-measurable such that $f = g$ $\overline{\mu}$-a.e..

*Proof.* If $f = \chi_E$ for $E \in \overline{\mathcal{M}}$ then $E = E_1 \cup F$ for $E_1 \in \mathcal{M}$ and $\overline{\mu}(F) = 0$. Then let $g := \chi_{E_1}$ and we are done, so we obtain the result for any simple $f$.
Let $\phi_n \to f$ point wise with $\phi_n$ simple and $\mathcal{M}$-measurable. Then our $\psi_n = \phi_n$ except on $E_n \in \overline{\mathcal{M}}$ with $\overline{\mu}(E_n) = 0$. Note that $\overline{\mu}(\cup E_n) = 0$ since there is $N$ null such that $\cup E_n \subset N$. So let $g = \lim_{n\to\infty} \psi_n \chi_{N^C}$ which is $\mathcal{M}$-measurable and $g = f$ except on $N$. □

Let $(X, \mathcal{M}, \mu)$ and define $L^+ := \{f : X \to [0, \infty], \text{measurable}\}$. If $\phi \in L^+$ is simple then $\phi = \sum_{i=1}^{\infty} c_i \chi_{E_i}$ and define:

$$\int \phi d\mu := \sum_{i=1}^{\infty} c_i \mu(E_i)$$

with the convention that $0 \cdot \infty = 0$. If $A \in \mathcal{M}$ then $\int_A \phi d\mu = \int \phi \chi_A d\mu$.

**Proposition 1.3.16**

Given $\phi$, $\psi \in L^+$ we have:

1. $\int c\phi d\mu = c \int \phi d\mu$ for $c \geq 0$;
2. $\int \phi + \psi d\mu = \int \phi d\mu + \int \psi d\mu$;
3. if $\phi \leq \psi$ then $\int \phi d\mu \leq \int \psi d\mu$;
4. for $A \in \mathcal{M}$ define $\mu_\phi(A) := \int_A \phi d\mu$ is a measure on $\mathcal{M}$.

**Definition 1.3.17**

Given $f \in L^+$ we define:

$$\int f d\mu := \sup\{\int \phi d\mu \mid \phi \leq f\}$$

Obviously if $f \leq g$ we have $\int f d\mu \leq \int g d\mu$ and also that $\int cf d\mu = c \int f d\mu$ and $\int f + g d\mu = \int f d\mu + \int g d\mu$.

**Theorem 1.3.18: Monotone Convergence Theorem**

f $\{f_n\} \subset L^+$ and $f_n \to f$ monotony increasing then the integral $\int f d\mu = \lim_{n\to\infty} \int f_n \, d\mu \iff \int \lim f_n d\mu = \lim \int f_n d\mu$.

*Proof.* Let $c_n := \int f_n d\mu$, then $c_n \leq c_{n+1}$ so $c_n \to c$. As $f_n \leq f$ and $f$ is measurable then $c_n = \int f_n d\mu \leq \int f d\mu$ so $c \leq \int f d\mu$.
Now let $0 < \alpha < 1$ and let $\phi$ be simple with $\phi \leq f$. Define $E_n := \{x | f_n(x) \geq \alpha\phi(x)\}$, then $\{E_n\} \subset X$ forms an expanding family such that $E_n \to X$.
Then we have that $c_n = \int_{E_n} f_n d\mu \geq \int_{E_n} \alpha\phi d\mu = \alpha\mu_\phi(E_n)$. By the limit theorem and taking the limit we have that:

$$\alpha\mu_\phi(\cup_{n=1}^{\infty} E_n) = \lim_{n\to\infty} \alpha\mu_\phi(E_n) = \alpha \int \phi d\mu \leq c$$

by taking the sup over $\phi$ we get that $c \geq \alpha \int f d\mu$ and so by taking the sup over $\alpha$ we get the other inequality and thus the required equality holds. □

**Corollary 1.3.19**

If $\{f_n\} \subset L^+$ then $\sum_{n=1}^{\infty} f_n \in L^+$ and $\int \sum_{n=1}^{\infty} f_n d\mu = \sum_{n=1}^{\infty} \int f_n d\mu$.

**Theorem 1.3.20**

If $f \in L^+$ then $\int f d\mu = 0 \iff f = 0$ $\mu$-a.e..

*Proof.* ($\Leftarrow$) Assume $f \equiv 0$ $\mu$-a.e. and let $0 \leq \phi \leq f$ be simple, then $\phi \equiv 0$ $\mu$-a.e. but then $\forall \phi : \int \phi d\mu = 0 \implies \int f d\mu = 0$.
($\Rightarrow$) Let $E_n = \{x | f(x) > \frac{1}{n}\}$ measurable, inverse of a measurable set. We then have that:

$$f \geq \frac{1}{n}\chi_{E_n} \implies 0 = \int f d\mu \geq \frac{1}{n}\mu(E_n) \implies \mu(E_n) = 0$$

so $E = \cup E_n = \{x | f(x) > 0\}$ implies that $\mu(E) = 0 \implies f = 0$ $\mu$-a.e.. □

**Lemma 1.3.21: Fatou**

f $f_n \in L^+$ then: $\int \liminf_{n\to\infty} f_n(x) d\mu \leq \liminf_{n\to\infty} \int f_n d\mu$.

*Proof.* Let $g_n := \inf_{k \geq n} f_k \leq f_j$ for $j \geq n$, then:

$$\lim_{n\to\infty} g_n = \liminf_{n\to\infty} f_n(x)$$

so we have $\forall k \geq n : \int g_n d\mu \leq \int f_k d\mu$ which implies $\int g_n d\mu \leq \inf_{k \geq n} \int f_k d\mu$, by monotone convergence we have:

$$\liminf_{n\to\infty} \int g_n d\mu = \int \lim_{n\to\infty} g_n d\mu \leq \lim_{n\to\infty} \inf_{k \geq n} \int f_k d\mu$$

which implies:

$$\int \liminf_{n\to\infty} f_n d\mu \leq \liminf_{n\to\infty} \int f_n d\mu$$

□

**Corollary 1.3.22**

If $f_n(x) \to f(x)$ $\mu$-a.e. then $\int f d\mu \leq \liminf_{n\to\infty} \int f_n d\mu$.

*Proof.* Let $f_n(x) \to f(x)$ on $E$ with $\mu(E^C) = 0$, then apply Fatou's Lemma on $f_n \chi_E$. □

**Corollary 1.3.23**

If $f \in L^+$, $\int f d\mu < \infty$ which implies $\mu(f^{-1}(\infty)) = 0$ and $\{x | f(x) > 0\}$ is $\sigma$-finite.

*Proof.* Let $E_n := \{x | f(x) \geq n\}$, then $f \geq n\chi_{E_n}$ which implies $\int f d\mu \geq n \int \chi_{E_n} d\mu = n\mu(E_n)$. So we have: $0 = \lim_{n\to\infty} \frac{1}{n} \int f d\mu \geq \mu(E_n)$ so $E = \{x | f(x) = \infty\} = \cap E_n$ has $\mu(E) = 0$. Then let $F_n = \{x | x > \frac{1}{n}\}$, $F = \{x | f(x) > 0\} = \cup F_n$, then $f(x) > \frac{1}{n}\chi_{F_n}$ which then $\infty > n \int f d\mu > \mu(F_n)$.
Hence we have that it is a countable union of finite measurable sets and hence $\sigma$-finite. □

### 1.3.3 Real and Complex Integration

Given $f : X \to [-\infty, \infty]$ if at least one of $\int f^+ d\mu$, $\int f^+ d\mu$ is finite we define:

$$\int f d\mu := \int f^+ d\mu - \int f^- d\mu$$

if both are finite we say that $f$ is integrable. Note that $|f| = f^+ + f^-$, then $f$ is integrable if and only if: $\int |f| d\mu < \infty$. If $f : X \to \mathbb{C}$, $f$ is integrable if and only if $\int |f| d\mu < \infty$. As $|f| \le |\mathfrak{Re} f| + |\mathfrak{Im} f| \le 2|f|$, so $f$ is integrable if and only if $\mathfrak{Re} f, \mathfrak{Im} f$ integrable. We define:

$$\int f d\mu = \int \mathfrak{Re} f d\mu + i \int \mathfrak{Im} f d\mu$$

we then have the following space:

$$L^1(X, \mu) := \{f : X \to \mathbb{C} \,|\, f \text{ measurable and } \int |f| d\mu < \infty\}$$

the space $L^1(X, \mu)$ is a complete vector space and $\int$ is a linear function on $L^1(X, \mu)$.

**Proposition 1.3.24**

For $f \in L^1 \implies |\int f d\mu| \le \int |f| d\mu$.

*Proof.* We have:

$$\begin{aligned} \left|\int f d\mu\right| &= \left|\int f^+ d\mu - \int f^- d\mu\right| \le \left|\int f^+ d\mu\right| + \left|\int f^- d\mu\right| \\ &= \int f^+ d\mu + \int f^- d\mu \\ &= \int |f| d\mu \end{aligned}$$

□

Given a complex valued function on $f : X \to \mathbb{C}$ we have for $\operatorname{sgn}(f) = \alpha$ ($\operatorname{sgn}(f) \cdot f = \frac{f^2}{|f|} \in \mathbb{R}$):

$$\left|\int f d\mu\right| = \overline{\alpha} \int f d\mu = \int \overline{\alpha} f d\mu = \int \mathfrak{Re} \overline{\alpha} d\mu \le \int |\overline{\alpha} f| d\mu = \int |f| d\mu$$

**Proposition 1.3.25**

If $f \in L^1$ than $\{x | f(x) \ne 0\}$ is $\sigma$-finite.

*Proof.* Decompose $f = u + iv = (u^+ - u^-) + i(v^+ - v^-)$ then apply $\sigma$-finite for real function in $L^+$ to $u^\pm$, $v^\pm$ to get the result. □

**Proposition 1.3.26**

the following are equivalent for $f, g \in L^1$:

1) $\forall E \in \mathcal{M} : \int_E f d\mu = \int_E g d\mu$;
2) $\int |f - g| d\mu = 0$;
3) $f(x) = g(x)$ $\mu$-a.e..

*Proof.* We have done (2) $\iff$ (3) already.
(2) $\implies$ (1) Let $\int |f-g|\,d\mu = 0$, then:

$$\left|\int_E f - g d\mu\right| = \left|\int (f-g)\chi_E d\mu\right| \le \int |f-g|\,\chi_E d\mu \le \int |f-g|\,d\mu = 0$$

(1) $\implies$ (2) Let $h := f - g = u + iv$. Assume one of $u^\pm$, $v^\pm$ has $E = \{x | u^+ > 0\}$ positive measure. The real part of the integral:

$$\mathfrak{Re}\int_E f d\mu - \int_E g d\mu = \mathfrak{Re}\int_E f - g d\mu = \int_E u^+ d\mu > 0$$

then we cannot have $\forall E : \int_E f d\mu = \int_E g d\mu$. □

So it does not make sense to alter functions on sets with measure zero. So we define the equivalence relations: $f \sim g \iff f(x) = g(x)$ $\mu$-a.e..
Note that $\{f : X \to \overline{\mathbb{R}},$ integrable$\}$ are a subset of $L^1(X,\mu)$ under $\sim$, moreover $\int$ is a linear function the space of equivalence classes.
Finally we have that $L^1(X,\mu)$ is a metric space with metric given by:

$$d(f,g) := \int_X |f-g|\,d\mu$$

under this equivalence relation we have that: $L^1(X,\mu) = L^1(X,\overline{\mu})$, $\overline{\mathfrak{m}}$ completion. Since this is a metric space we have $L^1$ convergence $f_n \to f$ in $L^1$ if $\int |f_n - f|\,d\mu \to 0$.

**Theorem 1.3.27: Lebesgue's Dominated Convergence Theorem**

Let $\{f_n\} \subset L^1$ and $f_n(x) \to f(x)$ $\mu$-a.e. and $0 \le f \in L^1$ such that $\forall n : |f_n| \le g$ $\mu$-a.e. then $f \in L^1$ and $\int f d\mu = \lim_{n\to\infty} \int f_n d\mu$.

*Proof.* Take $f = u + iv$ and $f_n = u_n + iv_n$, then $u_n \to u$ and $v_n \to v$ also $\forall n : |u|, |v| \le g$ $\mu$-a.e. so $u_n - g$ and $g + u_n$ are greater or equal than zero $\mu$-a.e., then by Fatou:

$$\int \liminf (g - u_n) d\mu \le \liminf \int g - \mu_n d\mu$$

which implies that:

$$\begin{aligned}\int \liminf (g-u_n)d\mu &\le \int g - \limsup \int u_n d\mu \\ \int g d\mu - \int u d\mu &\le \int g d\mu - \limsup \int u_n d\mu \\ \implies \int u d\mu &\ge \limsup \int u_n d\mu\end{aligned}$$

we also have that:

$$\begin{aligned}\int \liminf (g+u_n) \le \liminf \int g + u_n d\mu &= \int g d\mu + \liminf \int \mu_n d\mu \\ \int g d\mu + \int u d\mu &\le \int g d\mu + \liminf \int u_n d\mu\end{aligned}$$

finally:

$$\int \mu d\mu \le \liminf \int u_n d\mu \le \limsup \int u_n d\mu \le \int u d\mu$$

which yields:

$$\lim_{n\to\infty} \int u_n d\mu = \int \lim_{n\to\infty} u_n d\mu$$

and similarly for $v$. □

**Corollary 1.3.28**

If $f_n \in L^1$ and $\sum_{n=1}^\infty \int |f_n| d\mu < \infty$ then $\sum_{n=1}^\infty f_n \in L^1$ and $\int \sum_{n=1}^\infty f_n d\mu = \sum_{n=1}^\infty \int f_n d\mu$.

*Proof.* By monotone convergence:

$$\sum_{n=1}^\infty \int |f_n| d\mu = \int \sum_{n=1}^\infty |f_n| d\mu < \infty$$

so by $g := \sum_{n=1}^\infty |f_n| \in L^1 \implies \int g d\mu < \infty$ so $g < \infty$ $\mu$-a.e..
Then $g(x) = \sum_{n=1}^\infty |f_n(x)|$ is finite $\mu$-a.e. so we get that $\sum_{n=1}^\infty f_n(x)$ is absolutely convergent $\mu$-a.e..
Define $F_n(x) = \sum_{k=1}^n f_k(x) \in L^1$, $\forall n : |F_n(x)| \leq g(x)$, the by Lebesgue Dominated Convergence:

$$\lim_{n\to\infty} F_n = F \in L^1$$

so that:

$$\int F d\mu = \lim_{n\to\infty} \int F_n d\mu \iff \int \sum_{n=1}^\infty f_n d\mu = \sum_{n=0}^\infty \int f_n d\mu$$

□

**Corollary 1.3.29**

We have that $L^1(\mu)$ is a Banach space.

*Proof.* Let $f_n$ be a Cauchy sequence, so we have that:

$$\forall \epsilon > 0 : \exists n_\epsilon : \forall n, m \geq n_\epsilon : \int |f_n - f_m| d\mu < \epsilon$$

Take a subsequence $n_k$ such that the distance $\|f_{n_{k+1}} - f_{n_k}\| < \frac{1}{2^k}$. Let $g_k := f_{n_{k+1}} - f_{n_k}$, we then have that:

$$\sum_{i=1}^\infty \int |g_k| d\mu \leq \sum_{k=1}^\infty \frac{1}{2^k} < \infty$$

so by the previous corollary $G_n := \sum_{k=1}^N g_k \to G \in L^1$ then $\lim_{n\to\infty} f_{N+1} - f_1 = G \in L^1$ and in particular: $\lim_{N\to\infty} f_{n_N} = f_1 + G \in L^1$.
So the subsequence converges in $L^1$ so the whole sequence converges in $L^1$. □

**Theorem 1.3.30**

The simple function are dense in $L^1$.

*Proof.* Let $f \in L^1$ then there is $\phi_n \to f$ point wise and $|\phi_n(x)| \leq |f|$. Consider $|f - \phi_n| \leq 2|f| \in L^1$ so by Dominated Convergence $\lim_{n\to\infty} \int |f - \phi_n| d\mu = \int \lim_{n\to\infty} |f_n - \phi| d\mu = 0$ so we have that $\phi_n$ and $f$ are $L^1$ close.

□

Now let $\mu$ be a Lebesgue-Stieltjes measure on $\mathbb{R}$ for some monotone right-cotinuous function: $F$.

**Corollary 1.3.31**

For $\mu$ Lebesgue-Stieltjes the simple function over finite unions of open intervals are dense and also continuous (or smooth) functions with compact support are dense.

*Proof.* Let $f \in L^1(\mu)$ then there exists $\phi_n$ simple such that $\|f - \phi_n\| < \epsilon$ by the prior result. Assume $\phi_n = \sum_{i=1}^{M_n} c_k^n \chi_{E_k}^n$ for disjoint $E_k$ and $c_k^n \neq 0$. It follows that $\mu(E_k) < \infty$. By a previous result given $E \in \mathcal{M}$, $\mu(E) < \infty$ there exists $F$ finite union of open intervals such that $\mu(E \triangle F) < \epsilon$ for any $\epsilon$.
Choose $F_k^n$ fo $E_k^n$ such that $\mu(E_k^n \triangle F_k^n) < \frac{\epsilon}{|c_k^n| M_n}$. Consider the functions $\psi_n := \sum c_k^n \chi_{E_k^n}$, then:

$$\int |\psi_n - \phi_n| \leq \sum \int |c_k^n| \left|\chi_{E_k^n} - \chi_{F_k^n}\right| d\mu \leq \sum_{k=1}^{M_n} \int |c_k^n| \frac{\epsilon}{\left|c_k^n\right| M_n} < \epsilon$$

then by triangle inequality we get that:

$$\|f - \psi_n\| \leq 2\epsilon$$

Given any $\chi_{(a,b)}$ we can approximate it by either continuous or smooth function by using continuous or smooth bump functions so that: $\int \chi_E - \phi_n d\mu \leq 2\epsilon$. □

**Theorem 1.3.32**

Let $f : X \times [a,b] \to \mathbb{C}$ such that $f(-,t) : X \to \mathbb{C}$ is integrable for all $t$. For $F(t) := \int f(x,t) d\mu$ then:

(a) if $\forall x,t : |f(x,t)| \leq g(x) \in L^1$ and $\forall x : \lim_{t\to\infty} f(x,t) = f(x,t_0)$ then: $\lim_{t\to t_0} F(t) = F(t_0) \iff \lim_{t\to t_0} \int f(x,t_0) d\mu = \int \lim_{t\to t_0} f(x,t) d\mu$;

(b) if $\frac{\partial f}{\partial t}$ exists and $\left|\frac{\partial f}{\partial t}\right| \leq g(x) \in L^1$ then $F'(t) = \int \frac{\partial f(x,t)}{\partial t} d\mu$.

*Proof.* (a) For any $t_n \to t_0$ we have that $\lim_{n\to\infty} f(x,t_n) = f(x,t_0)$ let $f_n(x) := f(x,t_n) \in L^1$ also $|f_n(x)| \leq g \in L^1$ so the $\lim_{n\to\infty} \int f_n(x) d\mu = \int \lim_{n\to\infty} f_n(x) d\mu = \int f(x,t_0) d\mu$.
So we have that $\lim_{n\to\infty} F(t_n) = F(t_0) \implies \lim_{t\to t_0} F(t) = F(t_0)$.
(b) Let $h_n(x) := \frac{f(x,t_n) - f(x,t_0)}{t_n - t_0} \in L^1$, then $|h_n(x)| \leq \sup_{t\in[a,b]} \left|\frac{\partial f(x,t)}{\partial t}\right| \leq g(x)$ is is finite $\mu$-a.e. since $g$ is.
We then apply the Dominated Convergence Theorem:

$$\lim_{n\to\infty} \int h_n(x) d\mu = \int \lim_{n\to\infty} h_n(x) \implies \lim_{n\to\infty} \frac{F(t_n) - F(t_0)}{t_n - t_0} = \int \frac{\partial f(x,t_0)}{\partial t} d\mu$$

this is true for all sequence $t_n \to t_0$ hence $f$ differentiable at $t_0$ and $F'(t_0) = \int \frac{\partial f(x,t_0)}{\partial t} d\mu$. □

Consider $f : \mathbb{R} \to \mathbb{R}$ measurable with respect to the Lebesgue measure on $\mathbb{R}$. On $\mathbb{R}$ the Lebesgue measure is usually denoted by $(\mathbb{R}, \mathfrak{L}, \mathfrak{m})$ and is the unique Lebesgue measure with $\mathfrak{m}((a,b]) = b - a$. A function $f \in L^1(\mathfrak{m})$ is said to be Lebesgue Integrable.

### Riemann Integration

Let $f : [a,b] \to \mathbb{R}$ be a bounded function and take a partition $\mathfrak{p} : a = t_0 < t_1 < \ldots < t_n = b$, let $I_k := [t_{k-1}, t_k]$ and $M_k := \sup_{I_k} f$, $m_k := \inf_{I_k} f$. Let:

$$S_{\mathfrak{p}}(f) := \sum M_k \Delta_{t_j}, \quad s_{\mathfrak{p}}(f) := \sum m_j \Delta_{t_j}, \quad \Delta_{t_j} := t_j - t_{j-1}$$

then we define $\overline{I}_a^b(f) := \inf_{\mathfrak{p}\in\P} S_{\mathfrak{p}}(f)$ and $\underline{I}_a^b := \sup_{\mathfrak{p}\in\P} s_{\mathfrak{p}}(f)$. Then we say that $f$ is Riemann Integrable if:

$$\overline{I}_a^b(f) = \underline{I}_a^b$$

we then define $\int_a^b f dx := \overline{I}_a^b(f) = \underline{I}_a^b$.

**Theorem 1.3.33**

Let $f : [a, b] \to \mathbb{R}$ be bounded, then:

(a) if $f$ is Riemann Integrable then $f$ is Lebesgue with: $\int_{[a,b]} f d\mu = \int_a^b f dx$;

(b) the function $f$ is Riemann Integrable if and only if $f$ is discontinuous in a set of measure zero.

*Proof.* (a) Choose partitions $\mathfrak{p}_k$ such that $\mathfrak{p}_k \subset \mathfrak{p}_{k+1}$ and $S_{\mathfrak{p}_k}(f) \to \int_a^b f(x)dx$ and so does $s_{\mathfrak{p}_k}(f)$. Let $G_k := \sum_{j=1}^k M_j \chi_{(t_{j-1},t_j]}$ and $g_k := \sum_{j=1}^k m_j \chi_{(t_{j-1},t_j]}$ be simple functions and $|G_k|, |g_k| \leq M$ the bound of $f$. Hence, $G := \lim_{k\to\infty} G_k$ and $g := \lim_{k\to\infty} g_k$ are both measurable and in $L^1(\mathfrak{m})$.
Then from $\forall k : g_k \leq f \leq G_k$ we get that $g \leq f \leq G$. By Dominated Convergence: $\int F d\mathfrak{m} = \lim_{k\to\infty} \int G_k d\mathfrak{m} = \lim_{k\to\infty} S_{\mathfrak{p}_k}(f)$ and also $\int g d\mu = \lim_{k\to\infty} \int g_k d\mu = \lim_{k\to\infty} s_{\mathfrak{p}_k}(f)$ and they are both equal to $\int_a^b f dx$. So we have:

$$\int G d\mu = \int g d\mu = \int_a^b f dx \implies \int G - g d\mu = 0$$

but $G - g \geq 0 \implies G = g$ $\mu$-a.e. so $f = G = g$ $\mu$-a.e. as $\mathfrak{m}$ is a complete measure. Moreover, $\int f d\mu = \int f dx$.
(b) ($\Leftarrow$) Assume $f$ is discontinuous on a set $E$ with $\mu(E) = 0$. As before we have partitions $\{\mathfrak{p}_k\}$ increasing, so $\S_{\mathfrak{p}_k}(f) \to \overline{I}_a^b(f)$ and $s_{\mathfrak{p}_k}(f) \to \underline{I}_a^b(f)$. Let$F := \cup_{k\in\mathbb{N}} \mathfrak{p}_k$ and $N = E \cup F$ with $\mathfrak{m}(N) = 0$. So if $x \notin N$ we have that $\lim_{k\to\infty} G_k(x) = \lim_{k\to\infty} g_k(x)$ as $f$ is continuous at $x$, so in particular $G = g$ $\mu$-a.e.. Hence, $\int G f \mu = \overline{I}_a^b(f) = \int g d\mu = \underline{I}_a^b(f)$ hence $f$ is Riemann Integrable.

□

So proper Riemann Integrals are equal:

$$\int_a^b f(x)dx = \int_{[a,b]} f d\mathfrak{m}$$

but improper integration does not always hold. Improper absolutely convergent Riemann Integral are Lebesgue but general ones are not. So if we have absolute convergence: $\int_0^\infty f(x)dx = \int_{[0,\infty]} f d\mu$.

**Example.** Let $f := \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n} \chi_{(n-1,n]}$, then: $\int_0^\infty f dx = \lim_{t\to\infty} \int_0^t f dx = \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n} = \gamma < \infty$.
But since we have not absolute convergence we can rearrange it to get any number which does not make sense in the Lebesgue setting.

## 1.4 Product Measures

Given $(X, \mathcal{M}, \mu)$ and$(Y, \mathfrak{N}, \nu)$ measure spaces we want to produce a measure on $(X \times Y, \mathcal{M} \otimes \mathfrak{N})$, which we will call $\mu \times \nu$.
Define a premeasure for $A \times B \subset X \times Y$ with $A \in \mathcal{M}$ and $B \in \mathfrak{N}$ define $\pi(A \times B) := \mu(A)\nu(B)$ with the convention that $0 \cdot \infty = 0$.
Let $\mathcal{A}$ be the algebra of finite disjoint unions of rectangles $A \times B$ for $A \in \mathcal{M}$ and $B \in \mathfrak{N}$: $\mathcal{M}(\mathcal{A}) = \mathcal{M} \otimes \mathfrak{N}$ and extend $\pi$ to $\mathcal{A}$ by $\pi(\cup_{j=1}^n A_i \times B_i) = \sum_{i=1}^n \mu(A_i)\nu(B_i)$.

We need to prove that this is indeed a premeasure. Let $A \times B = \cup_j A_j \times B_j$ disjoint then $\chi_{A\times B}(\vec{x}) = \sum_j \chi_{A_j\times B_j}(\vec{x})$ which we can decompose for $\vec{x} = (x, y)$ as:

$$\chi_A(x)\chi_B(y) = \sum_j \chi_{A_j\times B_j}(x,y) = \sum_j \chi_{A_j}(x)\chi_{B_j}(y)$$

integrating with respect to $x$ and by the MCT we get that:

$$\mu(A)\chi_B(y) = \sum_j \mu(A_j)\chi_{B_j}(y)$$

which implies that $\pi(A \times B) = \sum_j \pi(A_j \times B_j)$ and so $\pi$ is a well defined premeasure. We then let $\mu \times \nu$ be the extension of $\pi$ to a measure on $\mathcal{M} \otimes \mathfrak{N}$.

**Remark.** If $\mu$ and $\nu$ are $\sigma$-finite then $\pi$ is $\sigma$-finite since if $X = \cup_j E_j$ monotonically with $\forall j : \mu(E_j) < \infty$ and $Y = \cup F_j$ monotonically with $\forall j : \nu(F_j) < \infty$ we get that:

$$\pi(E_j \times F_j) = \mu(E_j)\nu(F_j) < \infty \implies X \times Y = \cup_j E_j \times F_j$$

If $\mu$ and $\nu$ are $\sigma$-finite then $\mu \times \nu$ is the unique measure on $\mathcal{M} \otimes \mathfrak{N}$ with $\mu \times \nu(A \times B) = \mu(A)\nu(B)$.
Given $E \subset X \times Y$ we have cross sections at $(x, y)$:

$$E_x := \{y \in Y | (x,y) \in E\} \quad E_y := \{x \in X | (x,y) \in E\}$$

for $f$ a function on $X \times Y$ we have $f_x$ on $Y$ and $f^y$ on $X$ given by $f_x(y) := f(x,y) = f^y(x)$.

**Proposition 1.4.1**

1. If $E \in \mathcal{M} \otimes \mathfrak{N} \implies E_x \in \mathfrak{N}, E^y \in \mathcal{M}$;
2. If $f$ is $\mathcal{M} \otimes \mathfrak{N}$ measurable then $f_x$ is $\mathfrak{N}$-measurable and $f^y$ is $\mathcal{M}$-measurable.

*Proof.*

1. Let $\mathcal{R} := \{E \subset X \times Y | E_x \in \mathfrak{N}, E^y \in \mathcal{M}\}$, it is easy to show that this is a $\sigma$-algebra which implies that $\mathcal{M} \otimes \mathfrak{N} = \mathcal{M}(\mathcal{A}) \subset \mathcal{R}$ which then implies (1).
2. Let $f_x^{-1}(E) = [f^{-1}(E)]_x$ hence $f$ measurable implies $f^{-1}(E)$ measurable which implies $f^{-1}(E)_x \in \mathfrak{N}$ by (1) we get that $f_x$ is measurable and similarly for $f^y$.

□

**Definition 1.4.2**

A monotone class of subsets $\mathcal{C} \subset \mathcal{P}(X)$ in $X$ is a collection of subsets closed under increasing countable unions and decreasing countable intersections.
If $\mathcal{E} \subset \mathcal{P}(X)$ we define: $\mathcal{C}(\mathcal{E}) := \{\text{monotone class generated by } \mathcal{E}\}$ an elementary set.

**Lemma 1.4.3**

If $\mathcal{A}$ is an algebra then $\mathcal{M}(\mathcal{A}) = \mathcal{C}(\mathcal{A})$.

*Proof.* HWK □

**Theorem 1.4.4**

[Fubini-Tonelli] Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathfrak{N}, \nu)$ be $\sigma$-finite we then have that:

(a) if $f \in L^+(X \times Y)$ then $g(x) := \int f_x d\nu$, $h(y) := \int f^y d\mu$ are in $L^+(X)$, $L^+(Y)$ respectively with:

$$\int f d(\mu \times \nu) = \int g d\mu = \int h d\nu$$

(b) if $f \in L^1(\mu \times \nu)$ then $f_x \in L^1(\nu)$ a.e. in $x$ and $f^y \in L^1(\mu)$ a.e. in $y$, $g \in L^1(\mu)$ and $h \in L^1(\nu)$ with:

$$\int f d(\mu \times \nu) = \int g d\mu = \int h d\nu$$

*Proof.* (a) At first suppose that $\mu$, $\nu$ are finite. Let $\mathcal{C} := \{E | \chi_E \text{ satisfies (a)}\}$. Take $E_n$ monotonally increasing in $\mathcal{C}$ and let $E := \cup_{i=0}^{\infty} E_i$ by taking $f := \chi_{E_n}$ we get that $h_n(y) = \mu(E_n^y)$ is a measurable function as it is in $\mathcal{C}$ and satisfies (a). We then have that $h_n \le h_{n+1}$ and $h_n \to h$ with $h(y) = \mu(E^y)$, by Monotone Convergence we have:

$$\int h d\nu \overset{MC}{=} \lim_{n\to\infty} \overset{(a)}{=} \lim_{n\to\infty} (\mu \times \nu)(E) = \mu \times \nu(E)$$

by continuity from below of $\mu \times \nu$. Similarly for $g_n$.
We then have that $\chi_E$ satisfies (a) which implies that $E \in \mathcal{C}$ which means that $\mathcal{C}$ is closed under increasing unions.
Let $E_n \in \mathcal{C}$ with $E_{n+1} \subset E_n$ and let $E := \cap_n E_n$ then $h_n(y) = \mu(E_n^y)$ so $h_n \to h$ and $h(y) = \mu(E^y)$ by continuity from above.
Note that $h_n(y) \le \mu(X) < \infty$ and $\int \mu(X) d\nu = \mu(X)\nu(Y) < \infty$ so $\mu(X) \in L^1(\nu)$ so by DC we get that:

$$\int h d\nu = \lim_{n\to\infty} \int h_n d\nu = \mu \times \nu(E)$$

which implies that $E \in \mathcal{C}$ and $\mathcal{C}$ is closed under intersection hence $\mathcal{M} \otimes \mathfrak{N} = \mathcal{M}(\mathcal{A}) = \mathcal{C}(\mathcal{A}) \subset \mathcal{C}$ hence (a) is true for all $\chi_E \in \mathcal{M} \otimes \mathfrak{N}$.
If $X$, $Y$ are $\sigma$-finite then $X \times Y = \cup_{i=1}^{\infty} X_i \times Y_i$ finite and increasing. Define $E_i = E \cap (X_i \times E_i)$ apply to $\chi_{E_i}$ on $X_i \times Y_i$, then:

$$\chi_{E_i}(x, y) = \chi_{X_i}(x)\chi_{Y_i}(y)\chi_E(x, y)$$

and:

$$\mu \times \nu(E_i) = \int \chi_{X_i}(x)\nu(E_x \cap Y_i) d\mu$$

by Monotone Convergence of the RHS we get:

$$\mu \times \nu(E) = \int \nu(E_x) d\mu = \int \mu(E^y) d\nu$$

For $f \in L^+$ note that (a) is true for $f = \chi_E$ which implies true for simple function. We then have $\phi_n \le \phi_{n+1}$ with $\phi_n \to f$ and by Monotone Convergence we get the result for $f$.
(b) For $f \in L^1(\mu \times \nu)$ we prove it on $f^+, f^- \in L^+$ and $\int f^+ d(\mu \times \nu) < \infty$ and $\int f^+ d(\mu \times \nu) = \int h^+(y) d\nu$ hence $h^+(y) < \infty$ $y$-a.e. which means that $\int f^y d\mu < \infty$ $\nu$-a.e.. then combining the result for $f^+$ and $f^-$ we get it for $f = f^+ - f^-$.

□

**Remark.** The measure $\mu \times \nu$ is almost never complete since even if $(X, \mathcal{M}, \mu)$ and $(Y, \mathfrak{N}, \nu)$ are both complete we could have that for $A \subset Y$ not measurable[a] and for $N \in \mathcal{M}$ such that $\mu(N) = 0$ we have that $E := N \times A \notin \mathcal{M} \otimes \mathfrak{N}$ as $E_x = N$ but then $N \times A \subset N \times Y$ and so $0 \le \mu \times \nu(N \times A) \le \mu \times \nu(N \times Y) = \mu(N)\nu(Y) = 0$ hence

$\mu \times \nu$ is not complete.

[a] By Vitali we have such a set.

**Theorem 1.4.5**

[Fubini-Tonelli Complete measures] Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathfrak{N}, \nu)$ be complete and $\sigma$-finite. Let $(X \times Y, \mathfrak{L}, \lambda)$ be a completion of $(X \times Y, \mathcal{M} \otimes \mathfrak{N}, \mu \times \nu)$ we have that:

(a) if $f \in L^+(\lambda)$ then $g(x) := \int f_x d\nu$, $h(y) := \int f^y d\mu$ are in $L^+(X)$, $L^+(Y)$ respectively with:

$$\int f d\lambda = \int g d\mu = \int h d\nu$$

(b) if $f \in L^1(\lambda)$ then $f_x \in L^1(\nu)$ a.e. in $x$ and $f^y \in L^1(\mu)$ a.e. in $y$, $g \in L^1(\mu)$ and $h \in L^1(\nu)$ with:

$$\int f d\lambda = \int g d\mu = \int h d\nu$$

We define $(\mathbb{R}^n, \mathfrak{L}^n, \mathfrak{m}^n)$ the Lebesgue Measure on $\mathbb{R}^n$ as the completion of $(\mathbb{R}^n, \mathfrak{L}^{\otimes n}, \mathfrak{m}^{\otimes n})$ and is the same as the completion of $(\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n}, \mathfrak{m}^{\otimes n})$ by uniqueness of the completion.

**Theorem 1.4.6**

If $E \in \mathfrak{L}^n$ then:

1. $\mathfrak{m}(E) = \inf\{\mathfrak{m}(U) | E \subset U, U \in \tau_{\mathbb{R}^n}\} = \sup\{\mathfrak{m}(K) | K \subset E, \text{ compact}\}$;
2. $E = B_1 \cup N_1 = B_2 \setminus N_2$ for $B_i$ Borel sets and $N_i$ Lebesgue null sets;
3. if $\mathfrak{m}(E) < \infty$ there is a finite collection of rectangles with integral size such that $R = \cup_{i=1}^k R_i$ satisfying $\mathfrak{m}(E \triangle R) < \epsilon$. Moreover, we can take rectangles with disjoint interior and non zero side.

*Proof.* 1. By definition of outer measure there exists $T_j$ rectangles such that:

$$\sum_{j=1}^{\infty} \mathfrak{m}(T_j) < \mathfrak{m}(E) + \epsilon$$

Now approximate $T_j$ by open rectangles $T_j \subset U_j$:

$$\mathfrak{m}(U_j) \leq \mathfrak{m}(T_j) + \frac{\epsilon}{2^j}$$

then:

$$E \subset \cup U_j = U \in \tau_{\mathbb{R}^n}$$

such that:

$$\mathfrak{m}(U) \leq \sum \mathfrak{m}(U_j) \leq \sum \mathfrak{m}(T_j) + \frac{\epsilon}{2^j} \leq \mathfrak{m}(E) + 2\epsilon$$

Hence, $\mathfrak{m}(E) = \inf\{\mathfrak{m}(U) | E \subset U, U \in \tau_{\mathbb{R}^n}\}$. Then $\mathfrak{m}(E) = \sup\{\mathfrak{m}(K) | K \subset E, K \text{ compact}\}$ is as the 1-dimensional case.

2. Similarly.

3. We have $K \subset \cup_{j=1}^{N} R_j \subset U$ and if we let $R := \cup_{j=1}^{n} R_j$ we have:

$$\mathfrak{m}(E \triangle R) < 2\epsilon$$

Now subdivide the rectangles in $R$ such that they are all disjoint and such that they are all of the form $[a_i' \times b_i'] \times [a_i \times b_i]$.

□

**Theorem 1.4.7**

If $f : \mathbb{R}^n \to \mathbb{R}$ such that $f \in L^1(\mathfrak{m}^n)$ then for $\epsilon > 0$ there exists simple functions, $\phi = \sum a_j \chi_{R_j}$, with $R_j$ product of intervals such that:

$$\int |f - \phi| \, d\mathfrak{m} < \epsilon$$

*Proof.* Approximate $f$ by simple functions: $\phi = \sum c_i \chi_{E_i}$. Now approximate $E_i$ with $R_i = \cup_{j=1}^{n_i} R_j^i$ disjoint rectangles:

$$\mathfrak{m}(E_i \triangle R_i) < \frac{\epsilon}{N \max |c_i|}$$

then let $\psi := \sum_{i,j} c_i \chi_{R_j^i}$ and we are done. If $g$ is continuous just approximate $\chi_R$ by continuous bump functions or also by smooth simple functions. □

The Lebesgue measure is invariant under translations $T_a : \mathbb{R}^n \to \mathbb{R}^n$ for $a \in \mathbb{R}^n$ and $T_a(x) := x + a$.

**Theorem 1.4.8**

If $E \subset \mathfrak{L}^n$ then $T_a(E) \in \mathfrak{L}^n$ and if $f : \mathbb{R}^n \to \mathbb{C}$ is Lebesgue measurable then $f \circ T_a$ is Lebesgue measurable. Moreover, if $f \in L^+$ or $L^1$then $f \circ T_a$ is in $L^+$ or $L^1$ respectively.

*Proof.* Note that $T_a$ is a continuous function hence $T_a^{-1}(B)$ is a Borel set for a Borel set. Then $T_a(B) = B + a = T_{-a}^{-1}(B)$ is Borel, hence $T_a$ preserves Borel sets.
For $E$ a rectangle $\mathfrak{m}(T_a(E)) = \mathfrak{m}(E)$ by the one dimensional case, then by uniqueness of extensions $\mathfrak{m}(T_a(B)) = \mathfrak{m}(B)$ for all Borel sets.
In particular if $\mathfrak{m}(B) = 0$ then $\mathfrak{m}(T_a(B)) = 0$ for all Borel sets. If $E \in \mathfrak{L}^n$ then $E = B \cup N$ for $N \subset N_1$ a Borel null set, hence:

$$\mathfrak{m}(T_a(E)) = \mathfrak{m}(T_a(B) \cup T_a(N)) = \mathfrak{m}(T_a(B)) = \mathfrak{m}(B) = \mathfrak{m}(E)$$

since $T_a(N) \subset T_a(N_1)$ which is a Borel null set.
Given $f : \mathbb{R}^n \to \mathbb{C}$ Lebesgue Measurable, then:

$$(f \circ T_a)^{-1}(E) = T_a^{-1}(f^{-1}(E)) = T_a^{-1}(E_1 \cup N_1) = \underbrace{T_a^{-1}(E_1)}_{\text{Borel}} \cup \underbrace{T_a^{-1}(N_1)}_{\text{null set}}$$

hence is an element of $\mathfrak{L}$.
Note for $f = \chi_E$:

$$\int f \circ T_a = \int \chi_{T_a^{-1}(E)} = \mathfrak{m}(T_a^{-1}(E)) = \mathfrak{m}(E) = \int f$$

which implies true for simple functions by linearity. By monotonicity we obtain the result for $L^+$ functions and then for $f \in L^1$ by just applying to the positive and negative parts. □

### 1.4.1 Outer Inner Content

Consider the lattices $2^{-k}\mathbb{Z}^n \subset \mathbb{R}^n$ and let $Q_k$ be the collection of closed cubes with length $\frac{1}{2^k}$ and vertices $\frac{1}{2^k}\mathbb{Z}^n$. Given a set $E$ then define:

$$\underline{A}(E,k) := \{Q \in Q_k | Q \subset E\} \quad \overline{A} := \{Q \in Q_k | Q \cap E \neq \varnothing\}$$

which yields:

$$\underline{A}(E,k) \subset E \subset \overline{A}(E,k)$$

we then have:

$$\underline{A}(E) := \cup_k \underline{A}(E,k) \quad \overline{A}(E) := \cap_k \overline{A}(E,k) \implies \underline{A}(E) \subset E \subset \overline{A}(E)$$

are all Borel sets, then $\overline{K}(E) := \mathfrak{m}(\underline{A}(E))$ and $\overline{K}(E) := \mathfrak{m}(\overline{A}(E))$. If $\underline{K}(E) = \overline{K}(E)$ then $E$ is Lebesgue measurable.

**Lemma 1.4.9**

For $U \subset \mathbb{R}^n$ open then $U = \underline{A}(U)$ and $U$ is a countable union of cubes with disjoint intersection.

*Proof.* Since $\underline{A}(U) \subset U$ it is sufficient to prove $U \subset \underline{A}(U)$. Let $x \in U$ then there exists $\delta > 0$ such that $B_\delta(x) \subset U$, for $k >> 0$ there is $Q \in Q_k$ with $2^{-k} < \delta$ such that $x \in Q$ and $Q \subset B_\delta(x)$, then $x \in \underline{A}(U)$.
Now $\underline{A}(U,0) \cup \bigcup_{k=1}^{\infty}(\underline{A}(U,k) \setminus \underline{A}(U,k-1)) = \underline{A}(U)$ is a countable union of cubes all of which have disjoint interiors, and $U$ is a disjoint union of cubes so $\mathfrak{m}(U) = \underline{K}(U)$. □

This immediately tells us that the Lebesgue measure of any open set is equal to tits inner contant.

**Lemma 1.4.10**

Let $C$ be compact, then $\mathfrak{m}(C) = \overline{K}(C)$.

*Proof.* Compactness gives $C \subset Q = [-2^N, 2^N]^n$ for some $N >> 0$. So we have that:

$$]\forall k : \overline{A}(C,k) + \underline{A}(Q \setminus C, k) = Q \implies \mathfrak{m}(\overline{A}(C,k)) + \mathfrak{m}(\underline{A}(Q \setminus C, k)) = \mathfrak{m}(Q)$$

so we have:

$$\lim_{k\to\infty}(\mathfrak{m}(\overline{A}(C,k)) + \mathfrak{m}(\underline{A}(Q \setminus C, k))) = \mathfrak{m}(Q)$$

and so:

$$\overline{K}(C) + \mathfrak{m}(Q \setminus C) = \mathfrak{m}(Q) \implies \overline{K}(C) = \mathfrak{m}(C)$$

□

**Lemma 1.4.11**

If $T \in GL_n(\mathbb{R})$ and $f$ is Lebesgue measurable then $f \circ T$ is Lebesgue measurable and for $f \in L^+(\mathfrak{m})$ or $L^1(\mathfrak{m})$, we have:

$$\int f d\mathfrak{m} = |\det T| \int (f \circ T) d\mathfrak{m}$$

*Proof.* Consider the elementary linear maps: $T_1(\vec{x}) = (x_1, \dots, cx_1, \dots, x_n)$, $T_2(\vec{x}) = (x_1, \dots, x_i + cx_j, \dots, x_n)$ and $T_3(\vec{x}) = (x_1, \dots, x_j, \dots, x_i, \dots, x_n)$.
Any $T \in GL_n(\mathbb{R})$ is a product of the above. Note that if $S, T$ satisfy the Lemma then:

$$\int f d\mathfrak{m} = |\det S| \int f \circ S d\mathfrak{m} = |\det S| |\det T| \int (f \circ S) \circ T d\mathfrak{m} = |\det(S \circ T)| \int f(S \circ T) d\mathfrak{m}$$

So we only need to prove the Lemma for $T_1$, $T_2$ and $T_3$. Assume $f$ is Borel measurable then $f \circ T$ is Borel measurable then by Fubini Theorem on $f = \chi_E$ Borel we get that the formula holds for $T_1$, $T_2$ and $T_3$ by changing the order of the integration. Moreover if $E$ is Borel since $T^{-1}$ is continuous we have that $T(E)$ is a Borel set, then by considering $f := \chi_{T(E)}$ we get: $\mathfrak{m}(T(E)) = |\det T| \mathfrak{m}(E)$. In particular Borel null sets are invariant under $T^{-1}$, $T$ and so is $\mathfrak{L}^n$.
By linearity we then get the result for simple functions, then by monotone convergence theorem we get it for $L^+$ function and then to $L^1$ functions. ☐

**Theorem 1.4.12**

Let $\Omega \subset \mathbb{R}^n$ be an open domain, $G : \Omega \to \mathbb{R}^n$ a $C^1$ diffeomorphism. If $f$ is Lebesgue on $g(\Omega)$ then $f \circ G$ is Lebesgue on $\Omega$ and $f \in L^+$, $f \in L^1$ implies:

$$\int_{G(\Omega)} f d\mathfrak{m} = \int_{\Omega} (f \circ G) |\det (DG)| d\mathfrak{m}$$

*Proof.* In $\mathbb{R}^n$ define $\|x\| := \max |x_j|$, for $T \in GL_n(\mathbb{R})$ define $\|T\| := \sup_{x \neq 0} \frac{\|T\|(x)}{\|x\|}$ we then have that: $\|T(x)\| \leq \|T\| \|x\|$. If $F$ is a $C^1$ diffeomorphism on a cube $Q$, then we have the following upper bound: $\mathfrak{m}(F(Q)) \leq \sup_{x \in Q} \|D_x F\|^n \mathfrak{m}(Q)$.
As $G$ is a $C^1$ diffeomorphism on a cube $Q$ then there is $\delta > 0$ such that:

$$\forall x, y : \|x - y\| < \delta : \left\|(D_x G)^{-1} D_y G\right\|^n \leq 1 + \epsilon$$

Decompose $Q$ into cubes $Q_j$ about points $x_j$ of size less than $\delta$:

$$\begin{aligned}
\mathfrak{m}(GQ) &\leq \sum_1^\infty \mathfrak{m}(GQ_j) \leq \sum_j \left|\det (D_{x_j} G)\right| \mathfrak{m}(DG_j^{-1}(GQ_j)) \\
&\leq (1 + \epsilon) \sum_j \left|\det (D_{x_j} G)\right| \mathfrak{m}(Q_j) \\
&\leq (1 + \epsilon)^j \int \left|\det (D_{x_j} G)\right| \chi_{Q_j} \quad \text{by DCT}
\end{aligned}$$

by $\epsilon \to 0$ we have:

$$\mathfrak{m}(GQ) \leq \int_Q |\det(D_x G)| d\mathfrak{m}$$

Now if $U$ is open we have that $U = \cup_{j=1}^\infty Q_j$ with disjoint interior. So:

$$\mathfrak{m}(G(U)) = \mathfrak{m}(G(\cup Q_j)) \leq \sum \mathfrak{m}(G(Q_j))$$

which gives:

$$\mathfrak{m}(G(U)) \leq \sum \int_{Q_j} |\det(D_x G)| d\mathfrak{m} = \int_U |\det(D_x G)| d\mathfrak{m}$$

Now for $E$ Borel and $\mathfrak{m}(E) < \infty$ we have $E \subset U_j$ open with $\mathfrak{m}(U_j) \leq \mathfrak{m}(E) + \frac{1}{j}$, then $V := \cap_{j=1}^\infty U_j$ and $\mathfrak{m}(V \setminus E) = 0$.

So:

$$\begin{aligned}\mathfrak{m}(G(E)) \le \mathfrak{m}(G(V)) &= \lim_{n\to\infty} \mathfrak{m}(G(\underbrace{\cap_{j=1}^n U_j}_{:=U_n})) = \\ &\le \lim_{n\to\infty} \int_{U_n} |\det(D_x G)| \qquad \text{by DCT} \\ &\le \int_V |\det(D_x G)| = \int_E |\det(D_x G)|\end{aligned}$$

so $\mathfrak{m}$ is $\sigma$-finite which implies that for a Borel set $E$ we can decompose it $E = \cup_j E_j$, with $E_j \subset E_{j+1} : \mathfrak{m}(E_j) < \infty$, then:

$$\begin{aligned}\mathfrak{m}(G(E)) = \mathfrak{m}(G(\cup_j E_j)) &= \mathfrak{m}(\cup(G(E_j))) = \\ &= \lim_{n\to\infty} \mathfrak{m}(G(E_j)) \le \lim_{j\to\infty} \int_{E_j} |\det(D_x F)| \\ &\le \int_E |\det(D_x G)|\end{aligned}$$

If $E$ is Lebesgue then $E = B \cup N$ with $N$ null, $N \subset N_1$ Borel null:

$$\mathfrak{m}(G(E)) \le \mathfrak{m}(G(B)) + \mathfrak{m}(G(N)) \le \int_B |\det(D_x G)| + 0$$

So:

$$\mathfrak{m}(G(E)) \le \int_B |\det(D_x G)| = \int_E |\det(D_x G)|$$

thus we proved inequality for $E \in \mathfrak{L}$:

$$\mathfrak{m}(G(E)) \le \int_E |\det(D_x G)|$$

so for $\phi = \chi_{G(E)}$ we have:

$$\int_{G(\Omega)} \phi \le \int_\Omega \phi \circ G\,|\det(D_x G)|$$

then by linearity is true for $\phi \in L^+$ simple and so for arbitrary measurable functions by monotonicity, so for $f \in L^+$:

$$\int_{G(\Omega)} f d\mathfrak{m} \le \int_\Omega f \circ G |\det(D_x G)|\, d\mathfrak{m} \le \int_{G(\Omega)} f \circ G \circ G^{-1} \left|\det(D_{G(x)} G)\right| \left|\det(D_x G^{-1})\right| = \int_{G(\Omega)} f d\mathfrak{m}$$

So we have equality for $f \in L^+$ and for $f \in L^1$ by linearity on $f^+, f^-$. □

**Overview of Integration**

We have the following:

- For $f : X \to Y$ and $(X, \mathcal{M}), (Y, \mathfrak{N})$ is $(\mathcal{M}, \mathfrak{N})$-measurable if and only if $f^{-1}(E) \in \mathcal{M} \forall e \in \mathcal{M}$.
- So $f : X \to \mathbb{R}$ or $(\overline{\mathbb{R}}, \mathbb{C})$ is $\mathcal{M}$-measurable if is $(\mathcal{M}, \mathcal{B})$-measurable for $\mathcal{B}$ the Borel sets.
- For $f : X \to \overline{\mathbb{R}}_+$ or $(\mathbb{C})$ we can approximate by simple functions from below. For $f \in L^+(\mu)$ we have: $\int f d\mu := \sup_{0 \le \phi \le f} \int \phi d\mu$.

Theorems: The following are some of the fundamental theorem for integration:

1. Monotone Convergence Theorem: If $f_n \in L^+$ and $f_n(x) \to f(x)$ from below then $f \in L^+$ and $\int f_n d\mu \to \int f d\mu$.
2. Fatou: if $f_n \in L^+ : \int \liminf f_n \le \liminf \int f_n$.
3. Lebesgue Dominated Converge: for $f_n \in L^1(\mu)$ and $f_n(x) \to f(x)$ with $|f_n| \le g \in L^1$ then $f \in L^1$ and $f_n \xrightarrow{L^1} f$ and also $\int f_n \to \int f$.

4. Product Measure: for two measurable spaces $(X, \mathcal{M}, \mu)$ and $(Y, \mathfrak{N}, \nu)$ we have $\pi(A \times B) = \mu(A)\nu(B)$, $\pi$ is a pre-measure on the algebra $\mathcal{A}$ of disjoint unions of rectangles which extends to a measure by Carathéadory: then $(X \times Y, \mathcal{M} \otimes \mathfrak{N}, \mu \times \nu)$.

5. Fubini-Tonelli: If $f \in L^+(\mu \times \nu)$, $\mu, \nu$ are $\sigma$-finite then:

$$g(x) := \int f_x d\nu \quad h(y) := \int f^y d\mu \quad g, h \in L^1$$

and:

$$\int f d(\mu \times \nu) = \int g d\mu = \int h d\nu$$

if $f \in L^1(\mu \times \nu)$ and $\sigma$-finite, then $g, h \in L^1$ and:

$$\int f d(\mu \times \nu) = \int g d\mu = \int h d\nu$$

6. Lebesgue measure on $\mathbb{R}^n$: we have that $(\mathbb{R}^n, \mathfrak{L}^n, \mathfrak{m}^n)$ completion if $(\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n}, \mathfrak{m} \times \ldots \times \mathfrak{m})$ also the completion of: $(\mathbb{R}^n, \mathfrak{L} \otimes \ldots \otimes \mathfrak{L}, \mathfrak{m} \times \ldots \times \mathfrak{m})$.

**Proposition 1.4.13**

If $E \in \mathfrak{L}^n$ then

1. $\mathfrak{m}^n(E) = \inf\{\mathfrak{m}^n(U) | E \subset U \text{ open}\} = \sup\{\mathfrak{m}^n(K) | K \subset E \text{ compact}\}$;
2. $E = B_1 \cup N_1 = B_2 \setminus N_2$ for $B_i$-Borel and $N_i$ a null set.
3. $\mathfrak{m}(E) < \infty$ there exists a rectangles with disjoint interior and interval sides such that: $\mathfrak{m}(E \triangle \cup_1^n R_i) < \epsilon$.

*Proof.* Define $T_a(x) := x + a$ then $\mathfrak{m}(T_a(E)) = \mathfrak{m}(E)$ and $T \in GL_n(\mathbb{R})$, then:

$$\det T \int_E f \circ T = \int_E f$$

since $G : \Omega \to \mathbb{R}^n$ is a $C^1$ diffeomorphism:

$$\int_{G(\Omega)} f = \int_\Omega f \circ G |\det(D_x G)|$$

□

## 1.5 Signed Measure and Differentiation of Measures

The motivation is to compare two measures $\mu$, $\nu$ on $(X, \mathcal{M})$ does $\lim_{E \to \{x\}} \frac{\mu(E)}{\nu(E)}$ makes sense?

### 1.5.1 Signed Measure

**Definition 1.5.1**

A signed measure on $(X, \mathcal{M})$ is a function $\nu : \mathcal{M} \to [-\infty, \infty]$ such that it satisfies:

1. $\nu(\varnothing) = 0$;
2. $\nu$ only assumes at most one of $+\infty$ or $-\infty$;

3. $E_i \in \mathcal{M}$ are disjoint then $\nu(\cup E_i) = \sum_{i=1}^{\infty} \nu(E_i)$ and is absolute convergent.

if $\nu : \mathcal{M} \to [0, \infty]$ is positive, if $\nu : \mathcal{M} \to [-\infty, 0]$ we say is negative.

**Example.**

1. If $\mathfrak{m}_1$, $\mathfrak{m}_2$ are regular measures with one of them finite then: $\nu = \mathfrak{m}_1 - \mathfrak{m}_2$ is a signed measure;
2. If $f : X \to \overline{\mathbb{R}}$ is measurable with one of $\int f^+$, $\int f^-$ finite, then: $\nu(E) := \int_E f d\mu$ is a signed measure.

Observations: If $E \subset F \in \mathcal{M}$ and $\nu(E) = \pm\infty \implies \nu(F) = \pm\infty$ since:

$$\nu(F) = \nu(E) + \nu(F \setminus E) = \infty + \nu(F \setminus E) = \infty + c$$

for $c > -\infty$ since $\nu$ only attains one of the infinite.

**Proposition 1.5.2**

Let $\nu$ be a signed measure on $(X, \mathcal{M})$, then if $\{E_j\}$ is an expanding sequence in $\mathcal{M}$: $\nu(\cup E_j) = \lim_{j\to\infty} \nu(E_j)$. If $\{E_j\}$ is a shrinking sequence in $\mathcal{M}$ and $\nu(E_1) < \infty$ then $\nu(\cap E_j) = \lim_{j\to\infty} \nu(E_j)$.

*Proof.* Let $F_j = E_j \setminus E_{j-1}$ then $\cup E_j = \cup F_j$ and they are disjoint, so we have:

$$\nu(\cup E_j) = \nu(\cup F_j) = \sum_{j=1}^{\infty} \nu(F_j) = \sum_{j=1}^{\infty} (\nu(E_j) - \nu(E_{j-1})) = \lim_{n\to\infty} \sum_{j=1}^{n} (\nu(E_j) - \nu(E_{j-1})) \lim_{n\to\infty} \nu(E_n)$$

the other one is similar. □

**Definition 1.5.3**

A set $E \in \mathcal{M}$ is positive if for all $F \subset E$, $F \in \mathcal{M}$ and $\nu(F) \geq 0$. Similarly we define negative and null sets.

**Lemma 1.5.4**

1. If $E$ is positive and $F \subset E$ then $F$ is positive.
2. If $E_i$ are positive then $E = \cup_{i=1}^{\infty} E_i$ is positive.

*Proof.* 1. Obvious.

2. Take $F \subset E = \cup E_i = \cup F_i$ with $F_i \subset E_i$ disjoint. Then $F = \cup_{i=1}^{\infty} F \cap F_i$ each of them is positive so: $\nu(F) = \sum_{i=1}^{\infty} \nu(F \cap F_i) > 0$.

□

Same holds for negative and null sets.

**Theorem 1.5.5: Hahn Decomposition**

If $\nu$ is a signed measure on $(X, \mathcal{M})$ there exists positive and negative sets $P$, $N$ for $\nu$ with $P \cup N = X$ and $P \cap N = \varnothing$. Furthermore if $P'$ and $N'$ is another such pair satisfying this then: $P \triangle P', N \triangle N'$ are null.

*Proof.* Without loss of generality assume $\nu$ does not attain $+\infty$, let $\mathfrak{m} = \sup\{\nu(E) | E \text{ positive}\} < \infty$, then there exists $P_i$ such that $\nu(P_i) \to \mathfrak{m}$, let $P = \cup P_i$ which is positive and $\nu(P) = \mathfrak{m} < \infty$. Hence $P$ has maximal measure over all positive sets, look at $N = X \setminus P$ if Null we are done. If $E \subset N$ and $E$ positive with $\nu(E) > 0$ then $\mathfrak{m}$ was not maximal: $\nu(E \cup P) = \nu(E) + \nu(P) > \mathfrak{m}$ and $E \cup P$ is positive hence a contradiction.
So $N$ contains no positive sets of positive measure. So if $B \subset N$ and $\nu(B) > 0$ then there exists $C \subset B$ with $\nu(C) < 0$, now let $A := B \setminus C \subset B$, then $\nu(A) = \nu(B) - \nu(C) > \nu(B) > 0$.
If $N$ is non negative we can define a particular sequence of sets in $N$. Let $n_1$ be the smallest integer such that there is $B \subset N$ with $\nu(B) > \frac{1}{n_1}$. Let $A_1$ be such a set satisfying $\nu(A_1) \geq \frac{1}{n_1}$, assume $A_1, \dots, A_{j-1}$ are defined, then there exists some $B \subset A_{j-1}$, $\nu(B) > \nu(A_{j-1}) + \frac{1}{n_j}$, where $n_j$ is the smallest such integer.
Let $A_j$ satisfy the above, then $A := \cap_{j=1}^{\infty} A_j$ and $\infty > \nu(\cap_{j=1}^{\infty} A_j) = \lim_{n\to\infty} \nu(A_j) > \sum_{j=1}^{\infty} \frac{1}{n_j}$ then $n_j \to \infty$ as $j \to \infty$. But we can choose $B \subseteq A$ with $\nu(B) > \nu(A) + \frac{1}{n}$ for some smallest $n$. But $B \subset A_j \implies \forall j : n_j \leq n$, then:

$$\nu(A) + \frac{1}{n} < \nu(B) \leq \nu(A_j)$$

which implies that $\sum \frac{1}{n_j}$ diverges which is a contradiction.
Suppose we have another pair $P'$, $N'$ another decomposition, then:

$$P \setminus P' \subset P, \quad P \setminus P' = N' \setminus N \subset N'$$

so the difference is both positive and negative which is equivalent to null and similar for $P' \setminus P$. □

**Definition 1.5.6**

Given $\mu, \nu$ signed measure, they are mutually singular if $\exists E, F \in \mathcal{M}$ with $E \cup F = X$ and $E \cap F = \varnothing$ with $E$ null for $\mu$, $E$ null for $\nu$ and we write $\mu \perp \nu$.

**Theorem 1.5.7**

If $\nu$ is a signed measure then there exists a unique positive measure $\nu^+, \nu^-$ such that $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$.

*Proof.* Let $P$, $N$ be an Hahn decomposition of $\nu$. Then:

$$\forall F \in \mathcal{M} : \nu^+(E) := \nu(E \cap P) \quad \nu^-(E) := -\nu(E \cap N)$$

$\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$ with $E = P, F = N$ uniqueness let:

$$\nu = \mu^+ - \mu^- \quad \mu^+ \perp \mu^-$$

then there exists: $EmF : E \cup F = X$ and $E \cap F = \varnothing$, $\mu^+(F) = 0 = \mu^-(E)$ which implies $E, F$ are positive and negative sets for $\nu$. Then $P \triangle E$, $N \triangle F$ are null for $\nu$, so:

$$A \in \mathcal{M} : \mu^+(A) = \mu^+(A \cap E) = \nu(A \cap E) = \nu(A \cap P) = \nu^+(A)$$

Since $\nu = \nu^+ - \nu^-$ one of them has to be finite since signed measure attain either $+\infty$ or $-\infty$. □

**Definition 1.5.8**

We define for $\nu$ a signed measure the total variation $|\nu| = \nu^+ + \nu^-$. Note that for $E \in \mathcal{M}$ it is $\nu$-null if and only if $|\nu|(E) = 0$. Moreover we have the following:

$$\nu \perp \mu \iff |\nu| \perp \mu \iff \nu^+ \perp \mu, \nu^- \perp \mu$$

For $\nu = \nu^+ - \nu^-$ we have that: $\mu = |\nu|$ and $f = \chi_P - \chi_N$ we get:

$$\int_E f d\mu = \nu(E)$$

so for $\nu$ any signed measure then:

$$\nu = \int f d\mu$$

for $f = \chi_P - \chi_N$ and $\mu = |\nu|$.

**Definition 1.5.9**

Let $L^1(\nu) = L^1(\nu^+) \cap L^1(\nu^-)$ then $\int f d\nu = \int f d\nu^+ - \int f d\nu^-$ and $f$ is integrable if one of them is finite.

**Definition 1.5.10**

Given $\mu$, $\nu$ measures on $(X, \mathcal{M})$ $\nu$ is absolutely continuous with respect to $\mu$ if $\forall E$ such that $\mu(E) = 0$ then $\nu(E) = 0$ and we write $\nu \ll \mu$.

Similarly if $\nu$ is a signed measure and $\mu$ is a positive measure we have that: $\mu(E) = 0 \implies E$ $\nu$-null and $|\nu|(E) = 0$.
Note: $\nu \ll \mu$ if and only if $|\nu| \ll \mu$ if and only if $\nu^+, \nu^- \ll \mu$.
If $\nu \ll \mu$ and $\nu \perp \mu$ then $\nu = 0$. If $\nu \perp \mu$ then there is $E, F$ with $E \cup F = X$ with $E \cap F = \varnothing$ such that $\mu(E) = |\nu|(F) = 0$ since $\nu \ll \mu \implies |\nu|(E) = 0$ which implies $|\nu|(X) = 0$.

**Lemma 1.5.11**

Let $\nu$ finite, $\mu$ positive, then $\nu \ll \mu \iff$ given $\epsilon > 0$ there exists $\delta > 0$ such that $|\nu(E)| < \epsilon$ for $\mu(E) < \delta$.

*Proof.* ($\Longleftarrow$) Assume $\mu(E) = 0$ and pick $\epsilon = \frac{1}{n}$ then $\forall n : |\nu(E)| \le \frac{1}{n}$ which implies $|\nu(E)| = 0$ and apply this to $E \cap P$ and $E \cap N$:$\nu^+(E))\nu(E \cap P) \le |\nu|(E) = 0$ and $\nu^-(E) = \nu(E \cap N) \le |\nu|(E) = 0$ hence we have that $\nu(E) = 0$.
($\Longrightarrow$) As $|\nu(E)| < |\nu|(E)$ we only need to show it for $|\nu|$. Assume is not true, then there is $\epsilon > 0$ such that for $\delta = 2^{-n}$ there exists $E_n$ with:

$$|\nu(E_n)| > \epsilon, |\mu(E_n)| \le 2^{-n}$$

Let $F_k := \cup_{j=k}^{\infty} E_j$ and $F = \cap_{k=1}^{\infty} F_k$, then:

$$\forall k : \mu(F) \le \mu(F_k) \le \sum_{j=k}^{\infty} 2^{-j} = 2^{1-k} \implies \mu(F) = 0$$

which is a contradiction since $\forall k : \nu(F_k) \ge \epsilon$ and since $\nu$ is finite we have:$\nu(F) = \lim_{n\to\infty} \nu(F_k) \ge \epsilon$ so that is not true that $\nu \ll \mu$. $\square$

If $\mu$ has positive measure, then $f \in \mathfrak{L}^1(\mu)$. For any $\epsilon > 0$ there exists $\delta > 0$ such that $\left|\int_E f d\mu\right| \le \epsilon$ for all $\mu(E) < \delta$.
For signed measure if $f \in \mathfrak{L}^1(\mu)$ for $\mu$ a positive measure then $\nu(E) = \int_E f d\mu$ defines a signed measure, write $d\nu = f d\mu$. Note $f d\mu \ll \mu$.

**Lemma 1.5.12**

If $\nu, \mu$ are finite measures on $(X, \mathcal{M})$ then $\nu \perp \mu$ or $\exists \epsilon > 0$ and $E \in \mathcal{M}$ with $\mu(E) > 0$ nad $\nu \geq \epsilon\mu$ on $E$.

*Proof.* Consider the signed measure $\nu - \frac{1}{n}\mu$. We then have by the Hahn-Decomposition that $X = P_n \cup N_n$.
Let $P = \cup P_n$ and $N = \cap_n N_n = P^C$ as $\forall n : N \subset N_n$ we have that $N$ is negative for $\nu - \frac{1}{n}\mu$ for all $n$.
For $E \subset N$ we have $\nu(E) - \frac{1}{n}\mu(E) \leq 0$ so $\forall n : 0 \leq \nu(E) \leq \frac{1}{n}\mu(E)$ so $\forall E \subset N$ we have that $\nu(E) = 0$ so in particular $\nu(N) = 0$. If $\mu(P) = 0$ then $\nu \perp \mu$.
Otherwise $\mu(P) > 0$ implies that $\mu(P_n) > 0$ for some $n$ and $\mu(P) \leq \sum_n \mu(P_n)$ and $P_n$ is a positive set for $\nu - \frac{1}{n}\mu$, then pick $F \subset E$ with $\nu(F) \geq \frac{\mu(F)}{n}$ is the required positive set. □

**Theorem 1.5.13: Lebesgue-Radon-Nikodym**

Let $\nu$ signed and $\sigma$-finite and $\mu$ $\sigma$-finite on $(X, \mathcal{M})$. Then there exists unique $\sigma$-finite signed measures $\lambda, \rho$ on $(X, \mathcal{M})$ such that:

$$\nu = \lambda + \rho, \lambda \perp \mu, \rho = f d\mu, \quad f \text{ extended}^a\ \mu\text{-integrable } \mu\text{-a.e.}$$

[a]We have that $f : X \to \mathbb{R} \cup \{\infty\}$.
[a]We have that $f : X \to \mathbb{R} \cup \{\infty\}$.
[a]We have that $f : X \to \mathbb{R} \cup \{\infty\}$.
[a]We have that $f : X \to \mathbb{R} \cup \{\infty\}$.

*Proof.* First suppose that $\mu$ and $\nu$ are finite. Let:

$$\mathcal{F} := \{f : X \to [0, \infty] \text{ measurable} | \forall E \in \mathcal{M} : \int_E f d\mu \leq \nu(E)\}$$

since $f \equiv 0 \in \mathcal{F}$ so we have that $\mathcal{F} \neq \emptyset$. If $f, g \in \mathcal{F}$ then so is $\max(f, g)$ so let $h := \max(f, g)$:

$$\int_E h d\mu = \int_{\{f \leq g\} \cap E} h d\mu + \int_{\{f > g\} \cap E} h d\mu = \int_{E_1} f d\mu + \int_{E_2} g d\mu \leq \nu(E_1) + \nu(E_2) = \nu(E)$$

Let $m := \sup\{\int_X f d\mu | f \in \mathcal{F}\}$ with $m < \nu(X) < \infty$. Let $f_n \in \mathcal{F}$ such that $\int f_n d\mu \to m$ and let $g_n := \max\{f_1, \ldots, f_n\}$ then $g_n \leq g_{n+1}$ and they converge to $g = \sup f_n$. Thus:

$$\int f_n d\mu \leq \int g_n d\mu \implies m \leq \lim_{n \to \infty} \int g_n d\mu$$

Since $\{g_n\} \in \mathcal{F}$ we have that $\lim_{n \to \infty} \int g_n d\mu = m$, then by the MCT we get:

$$\int g d\mu = \lim_{n \to \infty} \int g_n d\mu = m$$

and also:

$$\int_E g d\mu = \lim_{n \to \infty} \int_E g_n d\mu < \nu(E)$$

so $g \in \mathcal{F}$.
Now let $d\lambda := d\nu - g d\mu$ for $g \in \mathcal{F}$, this is a positive measure. Now by the previous Lemma either $\lambda \perp \mu \implies d\nu = d\lambda + g d\mu$ and so $\nu = \lambda + \rho, \lambda \perp \mu$ and $\rho \ll \mu$, or there is $\epsilon > 0$ and $E \in \mathcal{M}$ with $\mu(E) > 0$ and $\lambda \geq \epsilon\mu$ on $E$. Now let $h := g + \epsilon\chi_E$, then:

$$(g + \epsilon\chi_E) d\mu = g d\mu + \epsilon\chi_E d\mu \leq g d\mu + d\lambda \leq d\nu \implies g + \epsilon\chi_E \in \mathcal{F}$$

But $\int g + \epsilon d\chi_E d\mu = \int g d\mu + \epsilon\mu(E) = m + \epsilon\mu(E) > m$ which is a contradiction, hence we are done.
For uniqueness if $\nu = \lambda' + \rho'$ then $\lambda - \lambda' \perp u$ and $\rho' - \rho \ll \mu$ but $\lambda - \lambda' = \rho' - \rho = 0$ so they are the same.

Now $g$ is well defined $\mu$-a.e. and $\rho := fd\mu = f'd\mu$ implies that:

$$\forall E : \int_E fd\mu = \int_E f'd\mu \implies \int_E (f - f')d\mu = 0$$

so $f = f'$ $\mu$-a.e..

For the $\sigma$-finite we have that $X = \cup E_j = \cup F_j$ for $E_j$ and $F_j$ disjoint finite sets for $\mu$ and $\nu$ respectively, by taking intersections we obtain a sequence $A_j$ such that $X = \cup A_j$ and $A_j$ are finite for both $\mu$ and $\nu$. We then define $\mu_j(E) := \mu(E \cap A_j)$ and $\nu_j(E) := \nu(E \cap A_j)$. By the above we have that $\forall j : d\nu_j = d\lambda_j + f_j d\mu_j$ with $\lambda_j \perp \mu_j$. Since $\mu_j(A_j^C) = \nu_j(A_j^C) = 0$ we have that $\forall j : \lambda_j(A_j^C) = \nu_j(A_j^C) - \int_{A_j^C} f_j d\mu_j = 0$ so we may assume that $f_j|_{A_j^C} = 0$. Then let $\lambda := \sum_j \lambda_j$ and $f := \sum_j f_j$ and we have that $d\nu = d\lambda + fd\mu$ with $\lambda \perp \mu$ by exercise 4.9 and $d\lambda$ and $fd\mu$ are $\sigma$-finite as desired. Uniqueness is the same.

If $\nu$ is a signed measure we apply all the above to $\nu^+$ and $\nu^-$ respectively. □

> **Remark.** This theorem tells us that given a signed $\sigma$-finite measure we can compare it to any given $\mu$ $\sigma$-finite and we get that:
>
> $$\nu = \lambda + fd\mu \quad \lambda \perp \nu \quad \rho := fd\mu \ll \mu$$

**Corollary 1.5.14**

If $\nu \ll \mu$ then $d\nu = fd\mu$ ($\nu(E) := \int_E fd\mu$) and $f$ is well defined $\mu$-a.e..

The map $f$ is called the Radon-Nykodim derivative and we write $\frac{\mathrm{d}\nu}{\mathrm{d}\mu} = f$, so:

$$d\nu = \left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right) d\mu \implies \nu(E) = \int_E d\nu = \int g \left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right) d\mu$$

**Proposition 1.5.15: Chain Rule**

If $\nu$ is a signed measure and $\sigma$-finite, $\mu$, $\lambda$ $\sigma$-finite positive measures, such that $\nu \ll \mu \ll \lambda$ then:

1. if $g \in L^1(\nu)$ then $g\frac{\mathrm{d}\nu}{\mathrm{d}\mu} \in L^1(\mu)$ and $\int gd\nu = \int g\frac{\mathrm{d}\nu}{\mathrm{d}\mu}d\mu$;
2. $\nu \ll \lambda$ and $\frac{\mathrm{d}\nu}{\mathrm{d}\lambda} = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}$ $\lambda$-a.e..

*Proof.* Consider $\nu^+, \nu^-$ separately and add. Assume $\nu, \mu, \lambda$ positive and $\sigma$-finite. Then:

1. For $E \in \mathcal{M}$ we have: $\nu(E) = \int_E d\nu = \int_E \frac{\mathrm{d}\nu}{\mathrm{d}\mu}d\mu = \int \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\chi_E d\mu$, then is true for simple positive function so it is true for $L^+(\nu)$ by MCT and then to $L^1(\nu)$ by linearity.
2. Transitivity is obvious since:

   $$\nu(E) = \int_E \frac{\mathrm{d}\nu}{\mathrm{d}\mu}d\mu \overset{(1)}{=} \int_E \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}d\lambda$$

   so we have:

   $$\nu(E) = \int_E \frac{\mathrm{d}\nu}{\mathrm{d}\lambda}d\lambda = \int \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}d\lambda$$

   hence we have that $\frac{\mathrm{d}\nu}{\mathrm{d}\lambda} = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}$ $\lambda$-a.e..

□

**Corollary 1.5.16**

If $\mu \ll \lambda$ and $\lambda \ll \mu$ then $\frac{d\mu}{d\lambda}\frac{d\lambda}{d\mu} = 1$ almost everywhere with respect to both $\mu$ and $\lambda$.

### 1.5.2 Complex Measures

A complex measure $\nu$ on $(X, \mathcal{M})$ is a function $\nu : \mathcal{M} \to \mathbb{C}$ such that:

1. $\nu(\varnothing) = 0$;
2. $\{E_i\} \subset \mathcal{M}$ disjoint then $\nu(\cup E_i) = \sum \nu(E_i)$ is absolutely convergent.

If $\nu$ is a finite signed measure then $\nu$ is complete. If $f$ in $L^1(\mu, \mathbb{C})$ then $f d\mu$ is complex.
If $\nu$ is complex then we can decompose it as $\nu = \nu_r + i\nu_i$ with $\nu_r$ and $\nu_i$ finite and signed. We then have that: $L^1(\nu) := L^1(\nu_r) \cap L^1(\nu_i)$ and $\int f d\nu := \int f\nu_r + i \int f d\nu_i$.
Now for $\nu$ and $\mu$ complex we have that $\nu \perp \mu$ if $\mu_r \perp \nu_r$, $\mu_r \perp \nu_i$, $\mu_i \perp \nu_r$ and $\mu_i \perp \nu_i$. If $\mu$ complex and $\nu$ is positive measure we have that: $\mu \ll \nu$ if $\mu_r \ll \nu$ and $\mu_i \ll \nu$.

**Theorem 1.5.17: Lebesgue-Radon-Nykodim**

Let $\nu$ signed and $\sigma$-finite and $\mu$ $\sigma$-finite on $(X, \mathcal{M})$. Then there exists unique $\sigma$-finite signed measures $\lambda, \rho$ on $(X, \mathcal{M})$ such that:

$$\nu = \lambda + \rho, \lambda \perp \mu, \rho = f d\mu, \quad f \text{ extended } \mu\text{-integrable } \mu\text{-a.e.}$$

(Extended means that we have $f : X \to \mathbb{R} \cup \{\infty\}$.)

*Proof.* Apply to $\nu_i$ and $\nu_r$ separately. ☐

Note that $\nu \ll \mu \implies d\nu = f d\mu$ for $f \in L^1(\mu)$ hence $\frac{d\nu}{d\mu} = f \in L^1(\mu)$.

#### Total Variation

Let $\nu$ be a complex measure.

**Definition 1.5.18**

We define $|\nu|$ by $d|\nu| = |f| d\mu$ where $\nu = f d\mu$

This is unique since, suppose that $d\nu = f_1 d\mu_1 = f_2 d\mu_2$, we can then compare the 2 measure by $\rho = \mu_1 + \mu_2 \implies \mu_1 \ll \rho, \mu_2 \ll \rho$. We also have that $d\nu = f_1 d\mu_1 = f_1 \frac{d\mu_1}{d\rho}$ and $d\rho = f_2 \frac{d\mu_2}{d\rho} d\rho$ hence: $f_1 \frac{d\mu_1}{d\rho} = f_2 \frac{d\mu_2}{d\rho}$ $\rho$-a.e..
Note that all $\mu_1$, $\mu_2$ are positive measures we have that $\frac{d\mu_1}{d\rho} \geq 0$. We have that:

$$|f_1| \frac{d\mu_1}{d\rho} = \left| f_1 \frac{d\mu_2}{d\rho} \right| = |f_2| \frac{d\mu_2}{d\rho} \quad \rho\text{-a.e.}$$

hence:

$$|f_1| d\mu_1 = |f_1| \frac{d\mu_1}{d\rho} d\rho \overset{(*)}{=} |f_2| \frac{d\mu_2}{d\rho} d\rho = |f_2| d\mu_2$$

where the (*) equality comes from the fact that the two expressions are the same $\rho$-a.e..
For the existence part it suffices to take $\mu := |\nu_r| + |\nu_i|$.

**Proposition 1.5.19**

For $\nu$ complex on $(X, \mathcal{M})$ we get:

1. $|\nu(E)| \le |\nu|(E)$;
2. $\nu \ll |\nu|$ and $\left|\frac{\mathrm{d}\nu}{\mathrm{d}|\nu|}\right| = 1$ $|\nu|$-a.e.;
3. $L^1(\nu) = L^1(|\nu|)$ and $|\int f d\nu| \le \int |f| d|\nu|$.

*Proof.* 1. We have $\nu = f d\mu$ and $|\nu| = |f| d\mu$, $\nu(E) = \int_E f d\mu = |\nu(E)| = \left|\int_E f d\mu\right| \le \int_E |d| \mu = |\nu|(E)$.

2. From (1) we get that $|\nu|(E) = 0 \implies \nu(E) = 0$ hence $d\nu = g d|\nu|$ for $g = \frac{\mathrm{d}\nu}{\mathrm{d}|\nu|}$ hence $d\nu = f d\mu \overset{RND}{=} f g d|\nu| = g|f| d\mu$, which implies:

$$f d\mu = g|f| d\mu \implies f = g|f| \quad \mu\text{-a.e.} \implies f = g|f| \quad |\nu|\text{-a.e.}$$

the $f > 0$ $|\nu|$-a.e. ad $d|\nu| = |f| d\mu$, thus we ge that $|g| = 1$ $|\nu|$-a.e. and so $\frac{\mathrm{d}\nu}{\mathrm{d}|\nu|} = 1$ $|\nu|$-a.e..

3. We first prove that $L^1(\nu) \subset L^1(|\nu|)$. If $h \in L^1(\nu)$ then we have that $h\frac{\mathrm{d}\nu}{\mathrm{d}|\nu|} \in L^1(|\nu|)$ so we ge thtat its absolute values in in $L^1(|\nu|)$ hence $|h| \in L^1(|\nu|)$ and so $h \le |h|$ is in $L^1(|\nu|)$.

We also have that:

$$\left|\int h d\nu\right| = \left|\int h \frac{\mathrm{d}\nu}{\mathrm{d}|\nu|}\right| \le \int |h| \left|\frac{\mathrm{d}\nu}{\mathrm{d}|\nu|}\right| d|\nu| = \int |h| d|\nu|$$

In order to prove $L^1(|\nu|) \subset L^1(\nu)$ we have that $h \in L^1(|\nu|) \implies |h| \in L^1(|\nu|)$ hence if $\nu = f d\mu$ we have that $|\nu| = |f| d\mu$ so we get that $|h||f| \in L^1(\mu)$ and so $|h| f \in L^1(\mu)$ which implies that:

$$\int |h| f_{r,i}^{\pm} d\mu < \infty$$

and so $|h| \in L^1(\nu_{r,i}^{\pm}) \implies |h| \in L^1(\nu)$.

□

**Lemma 1.5.20**

If $\nu_1$ and $\nu_2$ are complex measures we have that $|\nu_1 + \nu_2| \le |\nu_1| + |\nu_2|$ by relating them to some absolutely continuous measure.

*Proof.* Let $\nu_i := f g_i d\mu_i$ and $\rho = \mu_1 + \mu_2$, then $\nu_i = g_i d\rho$ with $\nu_1 + \nu_2 \ll \rho$ and $d|\nu_1 + \nu_2| = |g_1 + g_2| d\rho \le (|g_1| + |g_2|) d\rho = d|\nu_1| + d|\nu_2|$ since $\nu_1 + \nu_2 = (g_1 + g_2) d\rho$. □

We then have that:

$$\left|\int d f \nu\right| \le \int |f| d\nu$$

### 1.5.3 Lebesgue Differentiation Theorem

For $f \in L^1$ on $\mathbb{R}^n$ if $f$ is Lebesgue measurable and $\int_K f d\mathfrak{m} < \infty$ for $K$ bounded and measurable we say that $f$ is locally integrable and we write $f \in L^1_{loc}$. We then introduce the following:

**Definition 1.5.21**

For $r > 0$ we define:

$$A_r f(x) = \frac{1}{\mathfrak{m}(B(x,r))} \int_{B(x,r)} f(y) dy$$

**Lemma 1.5.22**

If $f \in L^1_{loc}$ then $A_r f(x)$ is continuous in $(x,r) \in \mathbb{R}^n \times \mathbb{R}$.

*Proof.* Note that $\mathfrak{m}(B(x,r)) = r^n \mathfrak{m}(B(0,1)) = cr^n$ is a continuous function. Let $S(x,r) = \{y | |y-x| = r\}$ be the sphere about $x$, then $\mathfrak{m}(S(x,r)) = 0$ by taking an $\epsilon$ neighbourhood, if $(r,x) \mapsto (r_0, x_0)$ then $x_{B(x,r)} \to x_{B(x_0,r_0)}$-a.e.. By dominated convergence we have that:

$$\lim_{(x,r)\to(x_0,r_0)} \int_{B(r,x)} f(y) dy = \lim_{(x,r)\to(x_0,r_0)} \int f(y) \chi_{B(r,x)} dy$$

then for $(r,x)$ near $(r_0, x_0)$ we get:

$$\chi_{B(r,x)} \leq \chi_{B(r_0+1,x_0)}$$

so $g(y) = f(y)\chi_{B(r_0+1,x_0)} \in L^1$, so by DCT we get that:

$$\lim_{(r,x)\to(r_0,x_0)} \int f(y)\chi_{B(r,x)} dy = \int f(y)\chi_{B(r_0,x_0)} dy$$

Hence:

$$\lim_{(r,x)\to(r_0,x_0)} A_r f(x) = \lim_{(r,x)\to(r_0,x_0)} \frac{1}{\mathfrak{m}(B(r,x))} \int_{B(r,x)} f(y) dy = \frac{1}{\mathfrak{m}(B(r_0,x_0))} \int_{B(r_0,x_0)} f(y) dy = A_{r_0} f(x_0)$$

thus we have continuity. □

**Definition 1.5.23**

We define the <u>Hardy-Littlewood maximal function</u> $Hf(x) := \sup_{r>0} A_r |f| (x)$ for $f \in L^1$.

**Lemma 1.5.24**

We have that $H|f|$ is measurable.

*Proof.* We have that $H|f|^{-1}(a,\infty) = \cup_{r>0}(A_r|f|)^{-1}(a,\infty)$ which is a union of countable open intervals and $A_r|f|$ is continuous which implies Borel. □

**Lemma 1.5.25**

Let $U \in \mathbb{R}^n$ open and $U = \cup_{i\in I} B_i$, for $B_i$ balls. If $c < \mathfrak{m}(U)$ there exists $B_1, \ldots, B_k$ disjoint such that:

$$\sum_1^k \mathfrak{m}(B_i) > 3^{-n} c$$

*Proof.* Let $K \subset U$ compact with $\mathfrak{m}(K) > c$ then there exists a finite cover $A_1, \ldots, A_n$ of $B_i$. Choose $B_i$ to be the largest such ball and keep going by induction to construct $\{B_j\}$ is the largest disjoint from $B_1, \ldots, B_{j-1}$.
If $A_i$ is not in the list, then it intersects one of them. Let $A_i$ intersect $B_j$ be the smallest such $j$, then $A_i$ does not intersect $B_1, \ldots, B_{j-1}$.
So the radius $A_i$ is less or equal than the radius of $B_j$ and $A_i \subset B(x_j, 3r_j)$ for $r_j$ the radius of $B_j$, then $K \subset \cup B(x_j, 3r_j)$, so we have:

$$c \le \mathfrak{m}(K) \le \sum \mathfrak{m}(B(x_j, 3r_j)) \le 3^n \sum \mathfrak{m}(B_j)$$

□

**Theorem 1.5.26: The Maximal Theorem**

There is a constant $c > 0$ such that for all $f \in L^1$ and all $\alpha > 0$:

$$\mathfrak{m}(\{x | Hf(x) > \alpha\}) \le \frac{c}{\alpha} \int |f| \, d\mathfrak{m}$$

*Proof.* Let $E_\alpha = \{x | Hf(x) > \alpha\}$, if $x \in E_\alpha \implies \exists r_x$ such that $A_{r_x} |f|(x) > \alpha$, so:

$$\frac{1}{\mathfrak{m}(B(r,x))} \int_{B(r,x)} |f| \, dx > \alpha$$

by the prior lemma there exists finite disjoint balls in $E_\alpha$ such that.

$$\sum_{i=1}^{k} \mathfrak{m}(B(x, r_x)) \ge 3^{-n} c$$

so we have:

$$c < 3^n \sum_{i=1}^{k} \mathfrak{m}(B(x, r_x)) \le \frac{3^n}{\alpha} \sum_{i=1}^{k} \int_{B(x_i, r_i)} |f| \, d\mathfrak{m} \le \frac{3^n}{\alpha} \int_{\mathbb{R}^n} |f| \, d\mathfrak{m}$$

So we have that $c \le \frac{3^n}{\alpha} \int |f| \, d\mathfrak{m}$, let $c \to \mathfrak{m}(E_\alpha)$ then:

$$\mathfrak{m}(E_\alpha) \le \frac{3^n}{\alpha} \int |f| \, d\mathfrak{m}$$

and so the result. □

**Theorem 1.5.27**

For $f \in L^1_{loc}$ then $\lim_{n \to \infty} A_r f(x) = f(x)$ a.e..

*Proof.* We only need to prove it for $f \in L^1$ with $f(x)$ and $|x| > N$ for $x \in \mathbb{R}$ with $|x| \le N$ replace $f$ by $f\chi_{[-]}$.
Approximate $f$ by $g$ continuous with compact support, then $\int |f - g| \, dx < \epsilon$.
As $g$ has compact support we have that $g$ is uniformly continuous and there exists $r > 0$ such that $|g(x) - g(y)| < \delta$ for $|x - y| < r$. If we look at $|A_r g(x) - g(x)|$ we get:

$$|A_r g(x) - g(x)| = \frac{1}{\mathfrak{m}(B_r)} \left| \int_{B_r} (g(y) - g(x)) dy \right| \le \frac{\delta \mathfrak{m}(B_r)}{\mathfrak{m}(B_r)} = \delta$$

So $A_r g(x) \to g(x)$ as $r \to 0$. Now consider:

$$\begin{aligned}
\limsup |A_r f(x) - f(x)| &= \limsup_{r \to 0} |(A_r f(x) - A_r g(x)) + (A_r g(x) - g(x)) + (g(x) - f(x))| \\
&\le \limsup_{r \to 0} |A_r (f - g)(x)| + |A_r g(x) - g(x)| + |g(x) - f(x)| \\
&\le H(f - g)(x) + |f - g| (x)
\end{aligned}$$

Let $E_\alpha = \{x | \limsup |A_r f(x) - f(x)| > \alpha\}$, $F_\alpha = \{x | |f-g|(x) > \alpha\}$. Now we get $E_\alpha \subset F_{\frac{\alpha}{2}} \cup \{x | H(f-g)(x) > \frac{\alpha}{2}\}$, then:

$$\epsilon \geq \int_{F_\alpha} |f(x) - g(x)| x \geq \frac{\alpha}{2} \mathfrak{m}(F_{\frac{\alpha}{2}})$$

So $\mathfrak{m}(F_{\frac{\alpha}{2}}) \leq \frac{2\epsilon}{\alpha}$, we then have by the 1.5.26:

$$\mathfrak{m}(E_\alpha) \leq \frac{2\epsilon}{\alpha} + \frac{2C}{\alpha} \int |f-g|\, dx \leq \frac{2\epsilon}{\alpha} + \frac{2C\epsilon}{\alpha}$$

so by letting $\epsilon \to 0$ we get $\mathfrak{m}(E_\alpha) = 0$.

Now we have that: $\mathfrak{m}(\{x | \limsup |A_r f(x) - f(x)| > \alpha\}) = 0$. Since we have that: $\forall x \notin E = \cup_{n \in \mathbb{Z}} E_{\frac{1}{n}}$ the statement holds since we have $\mathfrak{m}(E) = \mathfrak{m}(\{x | \lim_{x \to 0} A_r f(x) \neq f(x)\}) = 0$ we have the statement $\mathfrak{m}$-a.e. as requested. □

**Applications**

For $E$ a Lebesgue set, let $f = \chi_E$ then we have that:

$$\lim_{r \to 0} A_r f(x) = f(x) \quad \mathfrak{m}\text{-a.e.}$$

Pick $x \in E$ where limit exists, then:

$$\lim_{r \to 0} \frac{\mathfrak{m}(E \cap B(x,r))}{\mathfrak{m}(B(x,r))} = 1$$

**Definition 1.5.28**

We define the Lebesgue set for $f$ as the following:

$$L_f := \{x | \lim_{r \to 0} \frac{1}{\mathfrak{m}(B(x,r))} \int_{B(x,r)} |f(y) - f(x)|\, dy = 0\}$$

**Theorem 1.5.29**

For $f \in L^1_{loc}$ then $\mathfrak{m}(L_f) = 0$.

*Proof.* Let $c \in \mathbb{C}$ and $g(x) := |f(x) - c|$, so by 1.5.27 there exists $E_c \in \mathcal{M}$ with $\mathfrak{m}(E_c) = 0$ such that:

$$\forall x \notin E_c : \lim_{r \to 0} \frac{1}{\mathfrak{m}(B(x,r))} \int_{B(x,r)} |f(y) - c|\, dy = |f(x) - c|$$

Take $D$ a countable dense subset of $\mathbb{C}$ and take $E := \cup_{c \in D} E_c$, then $\mathfrak{m}(E) = 0$, if $x \notin E$ there exists some $c \in D$ with $|f(x) - c| \leq \epsilon$ so that $\forall y : |f(y) - f(x)| \leq |f(y) - c| + \epsilon$, then:

$$\limsup_{r \to 0} \frac{1}{\mathfrak{m}(B(x,r))} \int_{B(x,r)} |f(y) - f(x)|\, dy \leq \limsup_{r \to 0} \frac{1}{\mathfrak{m}(B(x,r))} \int_{B(x,r)} |f(y) - c|\, dy + \epsilon \leq |f(x) - c| + \epsilon \leq 2\epsilon$$

which implies the result. □

Note that:

$$\left| \frac{1}{\mathfrak{m}(B(x,r))} \int_{B(x,r)} (f(y) - f(x)) dy \right| \leq \frac{1}{\mathfrak{m}(B(x,r))} \int_{B(x,r)} |f(y) - f(x)|$$

so this is a stronger result than the one we had before.

Now let $\{E_r\}$ be Borel sets in $\mathbb{R}^n$ such that $E_r \subset B(x,r)$ and $\mathfrak{m}(E_r) \leq \alpha \mathfrak{m}(B(x,r))$ (note that: $E_r$ does not have to contain $x$), such a family is said to shrink nicely at $x$.

**Theorem 1.5.30**

[Lebesgue Differentiation Theorem] Let $f \in L^1_{loc}$ then $\forall x \in L_f$ and every $E_r$ shrinking nicely at $x$:

$$\lim_{r\to 0} \frac{1}{\mathfrak{m}(E_r)} \int_{E_r} |f(x) - f(y)|\, dy = 0$$

*Proof.* We have that:

$$\frac{1}{\mathfrak{m}(E_r)} \int_{E_r} |f(y) - f(x)|\, dy \le \frac{1}{\alpha \mathfrak{m}(B(x,r))} \int_{B(x,r)} |f(y) - f(x)|\, dy$$

□

**Corollary 1.5.31**

We have that: $\lim_{r\to 0} \frac{1}{\mathfrak{m}(E_r))} \int_{E_r} f(y)dy = f(x)$ $\mathfrak{m}$-a.e..

**Regular Borel Measure**

**Definition 1.5.32**

A measure $\nu$ on $\mathbb{R}^n$ is regular if:

1. $\forall K$ compact: $\nu(K) < \infty$;
2. $\forall E \in \mathcal{M} : \nu(E) = \inf\{\nu(U) | U \subset U \text{ open}\}$.

a signed or complex regular measure is regular if $|\nu|$ is.

A regular measure is automatically $\sigma$-finite.

**Theorem 1.5.33**

Let $\nu$ be a regular signed or complex Borel measure. Then considering $\mathfrak{m}$ we have that for almost all $x \in \mathbb{R}^n$:

$$\lim_{r\to 0} \frac{\nu(E_r)}{\mathfrak{m}(E_r)} = f(x)$$

where $E_r$ shrinks nicely at $x$ and $d\nu = d\lambda + f d\mathfrak{m}$ is a L-R-N decomposition.

*Proof.* (Folland) We have $\nu, \mathfrak{m}$ $\sigma$-finite so $d\nu = d\lambda + f d\mathfrak{m}$ with $\lambda \perp \mathfrak{m}$ and $f d\mathfrak{m} \ll \mathfrak{m}$. Note that $|\nu| = |\lambda| + |f|\, d\mathfrak{m}$ and $|\nu|$ regular implies that $\lambda$ and $f d\mathfrak{m}$ are regular, so $f \in L^1_{loc}$.
As $\lambda \perp \mathfrak{m}$ we have $A$ a Borel set such that:

$$\lambda(A) = \mathfrak{m}(A^c) = 0$$

Let $F_k := \{x \in A | \limsup_{r\to 0} \frac{\lambda(B(x,r))}{\mathfrak{m}(B(x,r))} > \frac{1}{k}\}$, we want to show that for all $k$ $\mathfrak{m}(F_k) = 0$. By regularity of $\lambda$ we have $\forall \epsilon$: $A \subset U_\epsilon$ with $\lambda(U_\epsilon) < \epsilon$. Since $U_\epsilon$ is open for each $x \in F_k$ we have a ball $B_x \subset U_\epsilon$ such that $\lambda(B_x) > \frac{1}{k}\mathfrak{m}(B_x)$, then by Lemma 1.5.25 if we define $V^k_\epsilon := \cup_{x \in F_k} B_x$ and $c < \mathfrak{m}(V^k_\epsilon)$ we have $x_1, \dots, x_N$ with $B_{x_1}, \dots, B_{x_N}$ disjoint and:

$$c < 3^n \sum_{1=1}^{N} \mathfrak{m}(B_{x_i}) \le 3^n k \sum_{i=1}^{N} \lambda(B_{x_i}) \le 3^n k \lambda(V^k_\epsilon) \le 3^n k \lambda(U_\epsilon) \le 3^n k\epsilon$$

we then have that $\forall k : \mathfrak{m}(F_k) \le \mathfrak{m}(V^k_\epsilon) \to 0$ hence the result follows. □

# Chapter 2

# Functional Analysis

## 2.1 Normed Spaces

**Definition 2.1.1**

Let $V$ be a vector space on $\mathbb{R}$ or $\mathbb{C}$, a semi-norm: $\|-\| : V \to \mathbb{R}$ is a function such that:

- $\forall \lambda \in \mathbb{R}(\mathbb{C}), \forall v \in V : \|\lambda v\| = |\lambda| \, \|v\|$;
- $\|v + w\| \leq \|v\| + \|w\|$;
- $\|0\| = 0$.

We say that a semi-norm is a norm if $\|v\| = 0 \iff v = 0$. If $\|-\|$ is a norm then $(V, \|-\|)$ is a normed vector space with induced metric: $d(x, y) := \|x - y\|$. If $(V, \|-\|)$ is complete we call it a Banach Space.

**Example.** The following are all normed spaces:

- $C^0(X, \mathbb{R})$ continuous functions on $X$ a compact metric space with $\|f\| := \sup_{x \in X} |f(x)|$.
- $L^1(\mu)$ Lebesgue integrable functions with norm $\|f\| = \int_X |f| \, d\mu$.

**Theorem 2.1.2**

The normed vector space $(V, \|-\|)$ is complete if and only if absolutely convergent series in $V$ are convergent.

*Proof.* ($\Longrightarrow$) Let $s_n := \sum_{i=1}^n v_i$ be absolutely convergent, then:

$$\forall m > n : \|s_n - s_m\| = \left\| \sum_{i=n+1}^{m} v_i \right\| \leq \sum_{i=n+1}^{m} \|v_i\| \to 0$$

hence there is $N_\epsilon$ such that for all $m, n > N_\epsilon$ we have that: $\sum_{i=n+1}^{m} \|v_i\| < \epsilon$ so it is a Cauchy sequence so $s_n \to s \in V$ by completeness.

($\Longleftarrow$) Let $v_i$ be a Cauchy sequence, then: $\|v_n - v_m\| \leq \frac{1}{2^i}$ for $n, m > N_i$. Choose a sub-sequence $v_{n_i}$ with:

$$\|v_{n_{i+1}} - v_{n_i}\| \leq \frac{1}{2^i}$$

then let $w_i := v_{n_{i+1}} - v_{n_i} \in V$, then we have that $s_n := \sum_{i=1}^{n} w_i$ is absolutely convergent since $\|\sum_{i=1}^{\infty} w_i\| \leq \sum_{i=1}^{\infty} \|w_i\| \leq \sum_{i=1}^{\infty} \frac{1}{2^i} = 1$. So $s_k \to s \in V$ but:

$$\lim_{k\to\infty} s_k = s \implies \lim_{k\to\infty} v_{n_{k+1}} - v_{n_1} = s \implies \lim_{k\to\infty} v_{n_k} = s + v_{n_1}$$

so $v_i$ has a converging sub-sequence and so it converges. □

## 2.2 Bounded Linear Operators

**Definition 2.2.1**

A linear map: $L : X \to Y$ between normed vector space is bounded if $\exists C \geq 0 : \forall x : \|L(x)\| \leq C\|x\|$.

**Definition 2.2.2**

The space $L(X,Y)$ of linear operators from $X$ to $Y$ is a normed vector space with norm: $\|T\| := \sup_{0\neq x\in X} \frac{\|T(x)\|}{\|x\|}$.

This is indeed a norm since $\|\lambda T\| = |\lambda|\,\|T\|$ and for all $x \in X$ we have that for operators $S$ and $T$:

$$\|(S+T)(x)\| \leq \|S(x)\| + \|T(x)\| \leq \|S\|\,\|x\| + \|T\|\,\|x\| = (\|S\| + \|T\|)\,\|X\|$$

so $\|S+T\| \leq \|S\| + \|T\|$. If $\|T\| = 0$ then $\forall x \in X : \|T(x)\| = 0 \iff T(x) = 0 \iff T \equiv 0$.

**Theorem 2.2.3**

We have that $L(X,Y)$ is a Banach space whenever $Y$ is.

*Proof.* Let $T_n$ be a Cauchy sequence, then for all $x$:

$$\|T_n(x) - T_m(x)\| \leq \|T_n - T_m\|\,\|x\|$$

so $\{T_n(x)\}$ is Cauchy in $Y$, then the following is well defined:

$$T(x) := \lim_{n\to\infty} T_n(x)$$

and $T$ is linear since taking limit and $T_n$ are.
Moreover we have that since $|\|T_n\| - \|T_m\|| \leq \|T_m - T_n\|$ the sequence: $\{\|T_n\|\}$ is Cauchy in $\mathbb{R}$ hence for all $x$:

$$\|T(x)\| \leq \|T_n(x)\| + \|T_n(x) - T(x)\| \leq \|T_n\|\,\|x\| + \|T_n(x) - T(x)\|$$

as $n \to \infty$ we get that: $\|T_n(x)\| \leq \lim_{n\to\infty} \|T_n(x)\|\,\|x\| < \infty$ so $T \in L(X,Y)$. Moreover:

$$\|T_n(x) - T(x)\| \leq \|T_n(x) - T_m(x)\| + \|T_m(x) - T(x)\| \leq \|T_n - T_m\|\,\|x\| + \|T_m(x) - T(x)\|$$

so by sending $m \to \infty$ we can write:

$$\|T_n(x) - T(x)\| \leq \epsilon\,\|x\|$$

hence: $\frac{\|T_n(x)-T(x)\|}{\|x\|} \leq \epsilon$ so $T_n \to T$ in $L(X,y)$. □

**Theorem 2.2.4**

Let $T : V \to W$ linear, $T$ bounded $\iff$ $T$ continuous at zero $\iff$ $T$ continuous.

*Proof.* 1) $\implies$ 3) $\to$ 2) If $T$ is bounded then: $\|T(x)\| \leq \|T\| \, \|x\|$ hence: $\|T(x) - T(y)\| \leq \|T\| \, \|x - y\|$ so it is continuous.
3) $\implies$ 2) If $T$ is continuous at $0$, then $T$ is bounded: $\exists \delta > 0 : \|T(x)\| \leq 1$ for all $x \leq \delta$, so: $\|T(y)\| = \left\| \frac{1}{\delta} T(\delta y) \right\| \leq \frac{1}{\delta}$. ☐

**Definition 2.2.5**

A map $f : X \to Y$ between metric spaces is Lipschitz if there is $C > 0$ such that: $\forall x, y : d(f(x), f(y)) \leq Cd(x,y)$, Lipschitz maps are always continuous.

**Definition 2.2.6**

Let $L : V \to \mathbb{R}$ (or $\mathbb{C}$) be a linear functional if $L \in L(V, \mathbb{R})$ ($L(V, \mathbb{C})$). Denote by $V^* := L(V, \mathbb{K})$ for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We have that $V^*$ is a Borel Space.

**Example.** If $f \in L^1(\mu)$ bounded, we define: $L_f : L^1(\mu) \to \mathbb{R}$ by: $L_f(g) := \int fg d\mu$, and $|L_f(g)| \leq \int |f||g| \, d\mu \leq M \int |g| \, d\mu \leq M \|g\|$.

**Definition 2.2.7**

A sub-linear functional is a map: $p : V \to \mathbb{R}$ such that:

$$\forall \lambda \geq 0 : p(\lambda x) = \lambda p(x) \qquad p(x+y) \leq p(x) + p(y)$$

**Example.** Any semi-norm.

The following is a Theorem about extending linear functional with control over the norm.

**Theorem 2.2.8: Hahn-Banach**

Given $X$ a real vector space, $p : X \to \mathbb{R}$ a sub-linear functional, $Y \subset X$ a sub-space such that $f : Y \to \mathbb{R}$ is a linear functional with $\forall y \in Y : f(y) \leq p(y)$. Then there is $F : X \to \mathbb{R}$ linear such that: $F\big|_Y \equiv f$ and $\forall x \in X : F(x) \leq p(x)$.

*Proof.* Suppose we are given some extension $g : Y_1 \to \mathbb{R}$ with $Y \subseteq Y_1$ and $g\big|_Y = f$ and $\forall y \in Y : g(y) \leq p(y)$ (note $g = f$ is a perfectly good extension). If $Y_1 \neq X$ we extend by taking $x \in X \setminus Y_1$. Build an extension $h : Y_1 \oplus \mathbb{R} x \to \mathbb{R}$ in the following way: for $y_1, y_2 \in Y$ we have: $g(y_1 + y_2) \leq p(y_1 + y_2)$ then:

$$g(y_1) + g(y_2) \leq p(y_1 - x) + p(y_2 + x) \implies g(y_1) - p(y_1 - x) \leq p(y_2 + x) - g(y_2)$$

hence:

$$\sup_{y \in Y_1} \{g(y) - p(y - x)\} \leq \inf_{y \in Y_1} \{p(y + x) - g(y)\}$$

then let $\alpha$ be between these two values, then:

$$g(y) - p(y-x) \leq \alpha \leq p(y+x) - g(y)$$

which yields:

$$g(y) \leq \alpha + p(y-x) \qquad g(y) \leq p(y+x) - \alpha$$

we then define $h(\lambda x + y) := g(y) + \lambda\alpha$, $h$ is linear by definition and $h\big|_Y = g$. For $\lambda \neq 0$ we have:

$$h(y + \lambda x) = \lambda h\left(\frac{y}{\lambda} + x\right) = \lambda\left(h\left(\frac{y}{\lambda}\right) + \alpha\right)$$

for $\lambda > 0$ we also have:

$$\lambda\left(h\left(\frac{y}{\lambda}\right) + \alpha\right) \leq \lambda p\left(\frac{y}{\lambda} + x\right) = p(y + \lambda x)$$

for $\lambda < 0$ we have:

$$-\lambda\left(h\left(\frac{y}{-\lambda}\right) + \alpha\right) \leq -\lambda p\left(\frac{y}{-\lambda} - x\right) + \lambda\alpha - \lambda\alpha = p(y + \lambda x)$$

Then define a partial ordering on the set of extensions:

$$S := \{g : Y_g \to \mathbb{R} \,|\, Y \subset Y_g, g\big|_Y \equiv f, \forall y \in Y_g : g(y) \leq p(y)\}$$

with $g_1 \leq g_2$ if $Y_{g_1} \subset Y_{g_2}$ and $g_2\big|_{Y_{g_1}} \equiv g_1$. The set $S \neq \varnothing$ since $f \in S$.
Any chain has a maximum hence by Zorn we have a maximal element: $g : Y \to \mathbb{R}$ with nothing bigger hence $Y = X$. ☐

**Theorem 2.2.9: Complex Hahn-Banach**

Let $X$ be a complex vector space and $p : X \to \mathbb{R}$ a semi-norm. For $Y \subset X$ a subspace with $f : Y \to \mathbb{C}$ a complex $\mathbb{C}$-linear functional satisfying $\forall y \in Y : |f(y)| \leq p(y)$, then there exists $F : X \to \mathbb{C}$ complex linear such that $\forall x \in X : |F(x)| \leq p(x)$.

*Proof.* We have that $f(x) = u(x) + iv(x)$ linear $f(ix) = if(x) = -v(x) + iu(x)$, so we get that $f(x) = u(x) - iu(ix)$ and $|f(x)| \leq p(x)$, so: $|u(x)| \leq p(x)$. So we can extend $u : Y \to \mathbb{R}$ to $U : X \to \mathbb{R}$ with $|U(x) - iU(ix)| \leq p(x)$, then define $F(x) = U(x) - iU(ix)$ is $|F(x)| = \alpha F(x)$ for $\alpha$ a unit complex number so $F(\alpha x) = U(\alpha x) \leq p(\alpha x) = p(x)$. ☐

**Theorem 2.2.10**

Let $X$ be a normed vector space:

1. if $Y \subset X$ closed subspace and $x \in X \setminus Y$ then $\exists f \in X^*$ with $f\big|_Y = 0$, $f(x) \neq 0$ and $\|f\| = 1$;
2. if $x \neq 0$ then there is $f \in X^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$;
3. given $x_1 \neq x_2$ then there is $f \in X^*$ such that $f(x_1) \neq f(x_2)$, so $X^*$ separate points of $X$;
4. a map $X \to X^*$ with $x \mapsto x(f) := f(x)$ is an isometry onto its image.

*Proof.* 1. Since $Y$ is closed let: $\delta := \inf_{y \in Y} \|x - y\| \geq 0$, let $Y' := Y \oplus \mathbb{C}x$ for $x \notin Y$. Then $f(y + \lambda x) = \lambda\delta$, so:

$$|f(y + \lambda x)| = |\lambda|\,\delta \leq |\lambda| \left\|x - \left(-\frac{y}{\lambda}\right)\right\| \leq \|y + \lambda x\|$$

so there is an extension $F$ by Hahn-Banach. Moreover: $\forall y \in Y : F(y) = 0$ and $F(x) = \delta$ and $|F(x)| \leq \|x\|$.

2. Let $Y = \{0\}$ and apply $(a)$, then $|F(x)| = \delta = \|x\| \implies \|F\| \le 1$.

3. Apply to $(x-y) \neq 0 \implies F(x-y) \neq 0 \implies F(x) \neq F(y)$.

4. Suppose that $\hat{x}(f)$ to the norm of: $|\hat{x}(f)| = |f(x)| \le \|f\|\,\|x\|$ which implies $|\hat{x}(f)| = \hat{x}(-)$.

   Let $f^* \in X^*$ defined by $f(x)(x)$, then:

$$|\hat{x}(f)| = |f(x)| = \|x\| \implies \|\hat{x}\| \ge |\hat{x}f^*| \implies \|\hat{x}\| = \|x\|$$

   so we have $x \to \hat{x}$ is an isometry.

□

**Definition 2.2.11**

A space $X$ is called reflective if $X \to X^{**}$ is bijective. If $X$ is not Banach this cannot happen, its completion is a Banach subspace.

**Definition 2.2.12**

A sub set $U \subset (X,d)$ is nowhere dense if $\overline{U}$ does not contain an open ball.

**Definition 2.2.13**

A metric space $(X,d)$ is meager if it is a countable union of nowhere dense sets.

**Theorem 2.2.14: Baire Category Theorem**

Let $X$ be a complete metric space, then:

1. if $\{U_i\}_{i\in\mathbb{N}}$ are open, dense then $\cap_{i=1}^{\infty} U_i$ is dense;

2. the space $X$ is not meager.

*Proof.* 1. Let $V$ be open in $X$, we want to show: $V \cap \cap_{i=1}^{\infty} U_i \neq \varnothing$. For all $i$ we have $U_i \cap V$ is open and non empty as $U_i$ is dense, then there is $B(x_0, r_0) \subset U_1 \cap V$ and $r_0 \le 1$. By induction define $(x_i, r_i)$ such that $B(x_{n-1}, r_{n-1}) \subset U_{n-1}$ and with $r_n < \frac{1}{2^n}$. We then have for $\{x_n\}$ that: $\forall n, m \ge N : \|x_n - x_m\| \le 2r_n \le \frac{2}{2^n} \implies$ Cauchy so we have $x_n \mapsto x \in X$. Also $x_n \in B(x_N, r_N)$ for $n \ge N \implies x_n \in \overline{B(x_N, r_N)}$, hence: $x_n \in U_n \cap B(x_0, r_0) \implies \forall n : x \in U_n \cap B(x_0, r_0)$ so $\forall n : x \in U_n \cap V \implies x \in \cap_{n=1}^{\infty} U_n \cap V$.

2. Assume $X$ meager then $X = \cup_{i=1}^{\infty} E_i$ nowhere dense so $\forall n : \overline{E}_n^C$ are open and dense. Then: $\cap_{i=1}^{\infty} \overline{E}_i^C$ is dense, so not empty, hence:

$$X = \cup_{n=1}^{\infty} E_n \subset \cup_{n=1}^{\infty} \overline{E}_n \implies \varnothing = (\cup_{n=1}^{\infty} \overline{E}_n)^C = \cap_{n=1}^{\infty} \overline{E}_n^C$$

□

**Definition 2.2.15**

A map $f : X \to Y$ is an open mapping if $U \in \tau_X \implies f(U) \in \tau_Y$.

**Theorem 2.2.16: Open Mapping Theorem**

If $T \in L(X,Y)$ for $X$, $Y$ Banach, $T$ surjective then $T$ is open.

*Proof.* We only need to show that $\exists r > 0$ such that $B(0,r) \subset T(B(0,1))$ since then by translation we ge the general result. We have that $X = \cup_{n=1}^{\infty} B_n$ and $B_n = B(0,n)$.
Since $T$ is surjective then $Y = \cup_{n=1}^{\infty} T(B_n) \implies \exists n : T(B_n)$ is not nowhere dense, moreover: $y \mapsto ny$ is a homeomorphism from $T(B_1)$ to $T(B_n)$ so $T(B_1)$ is not nowhere dense.

1. Show $r > 0 : B_r = B(0,r) \subset \overline{T(B_1)}$, $T(B_1)$ is not nowhere dense so $y_1 \in B(y_0, \frac{r_0}{2})$ such that $y_1 \in T(B_1)$ hence $y_1 = T(x_1)$ so $B(y_1, \frac{r_0}{2}) \subset \overline{T(B_1)}$. Then: $B(T(x_1), \frac{r_0}{2}) \subset \overline{T(B_1)} \subset \overline{T(B(x_1, 2))} = T(x_1) + \overline{T(B(0,2))}$ by translating by $x_1$ we get:
$$B(0, \frac{r_0}{2}) \subset \overline{T(B(0,2))} \implies B(0, \frac{r_0}{2}) \subset \overline{T(B(0,1))}$$
we know $B_r := B(0,r) \subset \overline{T(B_1)}$, similarly $B_{\frac{r}{2^n}} \subset \overline{T(B_{\frac{1}{2^n}})}$ (by linearity of $T$).
2. Choose $y \in B_{\frac{r}{2}}$, then $y \in \overline{T(B_{\frac{r}{2}})}$ so there is $y_1 = T(x_1) \in \overline{T(B_{\frac{r}{2}})}$ such that: $\|y - y_1\| \le \frac{r}{4}$, then by induction we have $x_n \in B_{\frac{1}{2^n}}$ such that: $\|y - \sum_{i=1}^{n} T(x_i)\| \le \frac{r}{2^{n+1}}$.

   Let $y_2 := y - y_1$, then $y_2 \in B_{\frac{r}{4}} \implies x_2 \in B_{\frac{1}{4}}$ so $\|y_2 - T(x_2)\| \le \frac{r}{8}$.

   If $x_n \in B_{\frac{1}{2^n}}$ then: $\|y - \sum_{i=1}^{n} T(x_i)\| \le \frac{r}{2^{n+1}}$ so for $x := \sum_{n=1}^{\infty} x_n \in X$ with $\|x\| \le 1$ which implies that $x \in B_1$. Then: $y = \sum_{n=1}^{\infty} T(x_n) \implies y \in T(B_1)$ and then $B_{\frac{r}{2}}(0) \subset T(B_1)$.

□

**Corollary 2.2.17**

If $T \in L(X,Y)$ is bijective for $X$ and $Y$ Banach spaces then $T$ is an homeomorphism.

*Proof.* If $T$ is bijective we can define $T^{-1}$ a linear function, then $(T^{-1})^{-1}(U) = T(U)$ is open by the open mapping theorem. □

Consider the graph of a function $f : X \to Y$ such that $\Gamma(f) \subset X \times Y$, with the graph defined as $\Gamma(f) := \{x, y \in X \times Y | y = f(x)\}$. If $f$ is continuous we have that $\Gamma(f)$ is closed, since $(x_n, y_n) \to (x,y)$ for $x_n \to x$ and $y_n \to y$ and $(x_n, y_n) \in \Gamma(f)$ implies that $y_n = f(x_n)$ so continuity implies that $y = f(x)$ and so $(x,y) \in \Gamma(f)$.

**Corollary 2.2.18**

Let $T : X \to Y$ be a linear map with $\Gamma(f)$ closed and $X, Y$ Banach Spaces. Then $T \in L(X,Y)$.

*Proof.* If $\Gamma(f)$ is closed in $X \times Y$ then $\Gamma(f)$ is a Banach space with norm: $\|(x,y)\| := \max(\|x\|, \|y\|)$. Consider $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ and restrict them to $\Gamma(f)$. We then have:
$$\|\pi_1(x,y)\| = \|x\| \le \|(x,y)\| \implies \|\pi_1\| \le 1$$
similarly for $\pi_2$. So they are both bounded linear operators, We also have that $\pi_1$ is bijective when restricted to $\Gamma(f)$ hence is an homeomorphism, so $\|\pi_1^{-1}\| \le M$ by the above corollary, so $\pi_2 \circ \pi_1 : X \to Y$ is bounded, but this is exactly $T$, hence $T$ is bounded. □

**Theorem 2.2.19: Uniform Boundedness Theorem**

Let $X, Y$ be normed vector spaces and $\mathcal{A} \subset L(X, Y)$ a collection of linear maps from $X \to Y$. This:

1. If $\sup_{T\in\mathcal{A}} \|T(x)\| < \infty$ on a non meager-set then $\sup_{T\in\mathcal{A}} \|T\| < \infty$;
2. If $X$ is Banach, then $\sup_{T\in\mathcal{A}} \|Tx\| < \infty$ for all $x \in X$ then: $\sup_{T\in\mathcal{A}} \|T\| < \infty$.

*Proof.* 1. Let $E_n := \{x | \sup_{T\in\mathcal{A}} \|Tx\| \leq n\} = \cap_{T\in\mathcal{A}} \{x| \|Tx\| \leq n\}$, $E_n$ is an intersection of closed sets hence is closed. Let $F$ be the non-meager subsets of $X$, i.e. $F$ is not the countable union of nowhere dense for some $n$. then $\exists x_0, r_0 : B(x_0, r_0) \subset E_n \implies \overline{B(x_0, r_0)} \subset E_n$. Let $x \in \overline{B(0, r_0)}$ then $x = (x - x_0) + x_0$ so:

$$\|Tx\| \leq \|T(x - x_0)\| + \|T(x_0)\| \leq n + n = 2n$$

since they are both in $E_n$, hence:

$$\forall T \in \mathcal{A} : \|x\| \leq r_0, \|Tx\| \leq 2n \implies \forall T \in \mathcal{A} : \|T\| \leq \frac{2n}{r_0} = \sup_{T\in\mathcal{A}} \|T\| < \infty$$

2. If $X$ is Banach then $X$ is not meager by the Baire Category Theorem gives the result.

□

We can restate the theorem by: if $\{T_i\}_{i\in I} \subset L(X, Y)$ with $X$ Banach, then point-wise bounded implies that the operator is bounded, so: $\sup_{i\in I} \|T_i x\| < \infty \implies \sup_{i\in I} \|T_i\| < \infty$.

## 2.3 Hilbert Spaces

**Definition 2.3.1**

Let $V$ be a vector space: $\langle - | - \rangle : V \times V \to \mathbb{C}$ a complex inner product, then $(V, \langle - | - \rangle)$ is a pre-Hilbert space. We have a norm by $\|x\| = \sqrt{\langle x | x \rangle}$.

**Proposition 2.3.2: Cauchy-Schwarz**

e have that $|\langle v | w \rangle| \leq \|v\| \|w\|$.

*Proof.* If $\langle v | w \rangle = 0$ we are done. Suppose $|\langle v | w \rangle| \neq 0$. Then let $\alpha \in \mathbb{C}$, $|\alpha| = 1$ such that $x := \alpha w$, then $\langle v | w \rangle \in \mathbb{R}^+$ so:

$$\langle v | \alpha w \rangle = \overline{\alpha} \langle v | w \rangle = |\langle v | w \rangle| \implies \overline{\alpha} = \frac{\overline{\langle v | w \rangle}}{|\langle v | w \rangle|} = \frac{|\langle v | w \rangle|}{\langle v | w \rangle}$$

then: $\langle v | w \rangle = |\langle v | w \rangle|$. Now let:

$$\begin{aligned} 0 \leq \|v - tx\|^2 = \langle v - tx | v - tx \rangle = \\ = \|v\|^2 - 2t \langle v | x \rangle + t^2 \|x\|^2 = \\ = \|v\|^2 - 2t |\langle v | w \rangle| + t^2 \|w\|^2 \end{aligned}$$

then:

$$(b^2 - 4ac) \leq 0 \implies 4\,|\langle v\,|\,w\rangle| \leq 4\,\|v\|^2\,\|w\|^2$$
$$\implies |\langle v\,|\,w\rangle| \leq \|v\|\,\|w\|$$

equality holds if $v = tx$ for some $t$ i.e. so $v = cw$ for some $c \in \mathbb{C}$ and so $v$ and $w$ are linearly dependant. □

**Proposition 2.3.3**

Hence: $\|x\| = \sqrt{\langle x\,|\,x\rangle}$ is a norm.

*Proof.* We have that: $\|x+y\|^2 = \langle x\,|\,x\rangle + 2\mathfrak{Re}\,\langle x\,|\,y\rangle + \langle y\,|\,y\rangle$ which yields:

$$\|x+y\|^2 = \|x\|^2 + 2\,\|x\|\,\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2$$

□

What is a difference between norms and inner products? Inner products gives a notion of smoothness. Any symmetric convex domain defines a norm by declaring that radius to be a vector of unit length. For inner product the unit ball is smooth since we have a notion of angles, through the C-S inequality.

**Definition 2.3.4**

A Hilbert space is a complete pre-Hilbert space.

**Example.** The following are all Hilbert spaces:

- $\mathbb{C}^n$ with $\langle (z_1,\ldots,z_n)\,|\,(w_1,\ldots,w_n)\rangle = \sum_{i=1}^n v_i\overline{w_i}$;
- $L^2(\mu) = \{A|\int f^2 d\mu < \infty\}$ and $\langle f\,|\,g\rangle = \int f\overline{g}d\mu$, since $\int f\overline{g}d\mu \leq \frac{1}{2}\left(\int f^2 d\mu + \int g^2 d\mu\right)$.
- $\ell^2(\mathbb{N}) = \{\{a_n\}_1^\infty|\sum(a_n)^2 < \infty\}$ with $\langle \{a_n\}\,|\,\{b_n\}\rangle = \sum_{n=1}^\infty a_n\overline{b}_n$.

Similarly for a set $A$ let $\ell^2(A) := L^2(A, \mathcal{P}(A), \mu_A)$ the counting measure.

**Proposition 2.3.5**

[Parallelogram Law] We have for a Hilbert Space $(X, \langle -|-\rangle)$:

$$\forall x, y \in X : \|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

**Definition 2.3.6**

We say that $v \perp w$ if $\langle v\,|\,w\rangle = 0$. If $\{v_i\}_1^n$ such that $v_i \perp v_j$ for $i \neq j$ and $\|\sum_1^n v_i\|^2 = \sum_1^n \|v_j\|^2$, since: $\langle \sum v_i\,|\,\sum v_j\rangle = \sum_{i,j}\delta_{ij}\,\langle v_i\,|\,v_j\rangle = \sum_i \langle v_i\,|\,v_i\rangle$.

**Theorem 2.3.7**

If $V$ is Hilbert and $W$ is a closed subspace then $V = W \oplus^{\perp} W^{\perp}$ so for all $x \in V$: $x = y + z$ with $y$ closest point of $W$ to $x$ and $z$ closest point of $W^{\perp}$ to $x$.

*Proof.* Let $\delta := \inf_{y \in W} \|x - y\|$, let $\{y_n\} \in W$ with $\|x - y_n\| \to \delta$. We want to show that $\{y_n\}$ is Cauchy. We have:

$$2\left(\|y_n - x\|^2 + \|y_n - x\|^2\right) = \|y_n - y_m\|^2 + \|y_n + y_m - 2x\|^2$$

so:

$$\|y_n - y_m\|^2 = 2\left(\|y_n - x\|^2 + \|y_n - x\|^2\right) - 4\left\|\frac{y_n + y_m}{2} - x\right\|^2 =$$
$$\leq 2\left((\delta^2 + \epsilon) + (\delta^2 + \epsilon)\right) - 4\delta^2 \leq 2\epsilon$$

So is Cauchy hence it converges $y_n \mapsto y \in W$ since $W$ is closed.
Let $z := x - y$ then $x = y + z$. We want to show that $z \in W^{\perp}$. Let $u \in W$ and assume $\langle z \,|\, u \rangle \in \mathbb{R}$ (in case multiply $u$ by a norm 1 element so that this hold). Define $f(t) := \|z + tu\|^2 \geq \delta^2$ but:

$$f(t) = \|z\|^2 + +2t\langle z \,|\, u \rangle + t^2 \|u\|^2 \geq \delta^2$$

hence $f(t)$ has a minimum in $t = 0 \implies f'(t) = 0$ which yields that $\langle z \,|\, u \rangle = 0$ and thus $z \in W^{\perp}$.
Uniqueness: if we have $x = y + z = y' + z'$ then: $y - y' = z' - z \in W \cap W^{\perp} = 0$ so $z = z'$ and $y = y'$.
If $z' \in W^{\perp}$ we have that: $\|x - z'\|^2 = \|x - z\|^2 + \|z - z'\|^2 \geq \|x - z\|^2$ so $z$ is the nearest point of $W^{\perp}$. □

**Theorem 2.3.8**

Let $V$ be Hilbert and $f : V \to V^*$ defined by $f(v) = \langle v \,|\, - \rangle$ i.e. $f(v)(w) = \langle v \,|\, w \rangle$ then $f$ is injective and an isometry.

*Proof.* If $f(v) \equiv 0 \iff \forall w \in V : \langle v \,|\, w \rangle = 0 \implies f(v)(v) = \langle v \,|\, v \rangle = 0 \implies v = 0$. As far as the isometry goes we have that: $\|f(y)\| = \sup_{\|x\|=1} \|\langle y \,|\, x \rangle\| \leq \|y\|$ by Scwartz but since the sup is attained by $x = y$ we have that $\|f(y)\| = \|y\|$ and is thus an isometry. □

**Theorem 2.3.9**

The above map is an isomorphism.

*Proof.* We only need to prove surjectivity. Assume $0 \neq f \in V^*$ and let $W := \ker f$, then $W$ is closed and $W \neq V$ hence: $V \cong W \oplus^{\perp} W^{\perp}$ by the previous theorem. Let $0 \neq z \in W^{\perp}$ and suppose that $\|z\| = 1$ we then define for all $x \in V$:

$$u := f(z)x - f(x)z$$

so $f(u) = 0 \implies u \in W$ and so $\langle u \,|\, z \rangle = 0$ then:

$$0 = \langle f(z)x - f(x)z \,|\, z \rangle = f(z)\langle x \,|\, z \rangle - f(x)\langle z \,|\, z \rangle \implies f(x) = f(z)\langle x \,|\, z \rangle$$

hence: $f(x) = \langle \overline{z} \,|\, x \rangle$ for $\overline{z} = f(z)z$ so it is surjective. □

We have just proved that $f : V \to V^*$ is a linear isometry (conjugate linear on $\mathbb{C}$). Hence Hilbert spaces are reflexives $V \cong V^* \cong V^{**}$.

We say that $\{x_\alpha\}_{\alpha\in A}$ is convergent if $\sum_{\alpha\in A} x_\alpha := \sup_{F\subset A} \sum_{\alpha\in F} x_\alpha < \infty$ for $F$ all finite subsets of $A$. Fact: If $\sum_{\alpha\in A} x_\alpha < \infty$ then $x_\alpha \neq 0$ only in a countable subset of $A$.

**Proposition 2.3.10: Bessel's Inequality**

If $\{x_\alpha\} \subset V$ a pre-Hilbert space we have for $\{u_\alpha\}$ orthonormal that:

$$\forall x \in V : \sum_{\alpha\in A} |\langle x \,|\, u_\alpha\rangle|^2 \leq \|x\|^2$$

*Proof.* Let $F \subset A$ finite then:

$$0 \leq \left\| x - \sum_{\alpha\in F} \langle x \,|\, u_\alpha\rangle u_\alpha \right\|^2 = \|x\|^2 - \sum_{\alpha\in F} |\langle x \,|\, u_\alpha\rangle|^2$$

hence:

$$\forall F : \sum_{\alpha\in F} |\langle x \,|\, u_\alpha\rangle u_\alpha|^2 \leq \|x\|^2 \implies \sum_{\alpha\in A} |\langle x \,|\, u_\alpha\rangle|^2 \leq \|x\|^2$$

□

In particular we have that $\langle x \,|\, u_\alpha\rangle \neq 0$ for only a countable subset.

**Theorem 2.3.11**

Let $V$ be pre-Hilbert and $\{u_\alpha\}_{\alpha\in A}$ be a orthonormal set, then the following are equivalent:

1. $\forall\alpha : \langle x \,|\, u_\alpha\rangle = 0 \implies x = 0$;
2. $\|x\|^2 = \sum_{\alpha\in A} |\langle x \,|\, u_\alpha\rangle|^2$;
3. $x = \sum_{\alpha\in A} \langle x \,|\, u_\alpha\rangle u_\alpha$ with convergence independent of the order.

*Proof.* 3) $\implies$ 2) As $\langle x \,|\, u_\alpha\rangle \neq 0$ for a countable subset we have: $\{u_{\alpha_i}\}_{i\in I}$, then:

$$0 \leq \|x\|^2 - \sum_{i=1}^{n} \langle x \,|\, u_\alpha\rangle = \left\| x - \sum_{i=1}^{n} \langle x \,|\, u_\alpha\rangle u_{\alpha_i} \right\|^2$$

and by (3) the RHS goes to zero. Hence:

$$\|x\|^2 = \sum_{\alpha\in A} |\langle x \,|\, u_\alpha\rangle|^2$$

2) $\implies$ 1) Let $\langle x \,|\, u_\alpha\rangle = 0$ then $\forall\alpha \overset{2)}{\implies} \|x\|^2 = 0 \implies x = 0$.

1) $\implies$ 3) We have: $\sum_{i=1}^{\infty} |\langle x \,|\, u_{\alpha_i}\rangle|^2 \leq \|x\|^2 \implies$ absolute convergent.

Let $x_n := \sum_{i=1}^{n} \langle x \,|\, u_{\alpha_i}\rangle u_{\alpha_i}$, then $\|x_n - x_m\|^2 = \sum_{i=n+1}^{m} |\langle x \,|\, u_{\alpha_i}\rangle|^2 = 0$ hence $\{x_n\}$ is Cauchy so $x_n \mapsto \overline{x}$. Note: $\forall\alpha : \langle x - \hat{x} \,|\, u_\alpha\rangle = \langle x \,|\, u_\alpha\rangle - \lim_{n\to\infty} \langle x_n \,|\, u_\alpha\rangle = 0$. Then by 1) $x - \hat{x} = 0 \implies x = \hat{x}$ and so: $x = \sum_{\alpha\in A} \langle x \,|\, u_\alpha\rangle u_\alpha$. □

If $\{u_\alpha\}$ is orthonormal and satisfies any of the previous is called an orthonormal basis.

**Proposition 2.3.12**

Every Hilbert space has an orthonormal basis.

*Proof.* Use inclusion to partially order orthonormal sets and apply Zorn. □

Note: If $\{u_\alpha\}$ is a basis: $x = \sum_{\alpha\in A} \langle x\,|\,u_\alpha\rangle\, u_\alpha$ so we have:

$$F : V \to \ell^2(A)$$
$$x \mapsto \{\langle x\,|\,u_\alpha\rangle\}_{\alpha\in A}$$

**Theorem 2.3.13**

Let $V$ be Hilbert, then: $V$ is second countable $\iff$ $V$ has a countable orthonormal basis. Furthermore all orthonormal basis are countable if one is.

*Proof.* $\Longrightarrow$) If $V$ is second countable $\exists\{v_n\}_1^\infty$ countable dense. Discard $v_n$ if $v_n \in \text{span}\{v_1,\dots,v_{n-1}\}$. Obtain $\{w_n\}_1^\infty$ countable. Make it orthonormal using Gram-Schmidt, so that $\{w_n\}$ is an orthonormal set.
Suppose $v \notin \text{span} w_n$ then we have that $\forall n : \langle v\,|\,w_n\rangle = 0$, then since $\langle v\rangle$ is a closed subspace of $V$ and $V$ decomposes as $V \cong \langle v\rangle \oplus \langle v\rangle^\perp$ now since $\langle v\rangle^\perp$ is closed we get that $\overline{\text{span}}\{w_n\} \subset \langle v\rangle^\perp$ which is a contradiction.
$\Longleftarrow$) Let $\{v_n\}$ be an orthonormal basis and let $S = \{\sum_{q_i\in Q\text{finite}} q_i v_i | q_i \in \mathbb{Q}[i]\}$. Then $S$ is dense and countable. Now let $x = \sum_i^\infty a_i v_i$ for $a_i \in \mathbb{C}$, then: $\|x\|^2 = \sum_1^\infty |a_i|^2 < \infty$ choose natural finite approximation: $x_i = \sum_i^n q_i v_i$ with $|a_i - q_i|^2 < \frac{1}{i^2}$ with $q_i \in \mathbb{Q}[i]$ $(x_n \to x)$.
Say $\{u_\alpha\}_{\alpha\in A}$ is another orthonormal basis and let $A_n = \{\alpha \in A | \langle v_n\,|\,w_\alpha\rangle \neq 0\}$ non coefficients of $v_n$ with respect to $\{u_\alpha\}_{\alpha\in A} \implies A_n$ countable hence $\cup_1^\infty A_n$ is countable.
So $A := \cup_1^\infty A_n$ as if $v_\alpha \notin \cup_1^\infty A_n \implies \forall n : \langle u\alpha\,|\,v_n\rangle = 0 \implies u_\alpha = 0$. □

**Definition 2.3.14**

If $V_1, V_2$ are Hilbert a unitary map is a map: $U : V_1 \to V_2$ invertible linear such that $\langle Ux\,|\,Uy\rangle = \langle x\,|\,y\rangle$.

**Theorem 2.3.15**

If $V$ is Hilbert and $\{u_\alpha\}_{\alpha\in A}$ orthonormal basis, then: $V \cong \ell^2(A)$ i.e. under a map: $x \to \hat{x} = \{\langle x\,|\,u_\alpha\rangle\}_{\alpha\in A}$.

*Proof.* Let $f : V \to \ell^2(A)$ given by $f(x) = \{\langle x\,|\,u_\alpha\rangle\}_{\alpha\in A}$ and $(f(x))_\alpha = \langle x\,|\,u_\alpha\rangle$. Then $\|x\|^2 = \sum_{\alpha\in A} |\langle x\,|\,u_\alpha\rangle|^2 = \|f(x)\|^2 \implies \|x\| = \|f(x)\|$ so $f$ is an isometry. If $a = \{a_\alpha\}_{\alpha\in A} \in \ell^2(A)$, let $x = \sum_{\alpha\in A} a_\alpha u_\alpha \implies f(x) = a$ so $f$ is surjective.
Moreover we have that $\langle f(x)\,|\,f(y)\rangle = \sum_{\alpha\in A} \langle x\,|\,u_\alpha\rangle \cdot \langle y\,|\,u_\alpha\rangle \overset{???}{=} \langle x\,|\,y\rangle \dots$ □

Note classification of Hilbert Spaces is equivalent to the classification of sets in terms of cardinality and so they are all $\ell^2(A)$.

## 2.4 $L^p$ spaces

**Definition 2.4.1**

Given $(X, \mathcal{M}, \mu)$ and $L^p := \{f \text{ measurable} | \int_X |f|^p \, d\mu < \infty\}$.

**Definition 2.4.2**

For $A$ a set $\ell^p(A) = L^p(A, \mathcal{P}(A), \mu_A)$ the counting measure. The elements are sequences such that $\{\sum_{\alpha\in A} |x_\alpha|^p < \infty\} = \ell^p(A)$.

**Definition 2.4.3**

Define: $\|f\|_p := \left(\int |f|^p \, d\mu\right)^{\frac{1}{p}}$, so $\|-\|_p : L^p \to [0, \infty)$ is well defined and $\|kf\|_p = |k| \, \|f\|_p$.

Now for $A$ a set $\ell^p(A) = L^p(A, \mathcal{P}(A), \mu_A)$ so that $\|(x_i)_{i\in A}\|_p = \left(\sum_{i\in A} x_i^p\right)^{\frac{1}{p}}$. We have that $L^p$ is a vector space, since:

$$|f + g|^p \le (2\max(|f|, |g|))^p \le 2^p(|f|^p + |g|^p)$$

implies that:

$$\int |f + g|^p \, d\mu \le 2^p \int |f|^p \, d\mu + \int |g|^p \, d\mu < \infty$$

**Definition 2.4.4**

We say that $p, q \ge 1$ are conjugate if $\frac{1}{p} + \frac{1}{q} = 1 \iff \frac{1}{q} = 1 - \frac{1}{p} = \frac{p-1}{p} \iff q = \frac{p}{p-1} \iff (p-1)q = p$.

**Proposition 2.4.5: Hölder Inequality**

If $\frac{1}{p} + \frac{1}{q} = 1$ and $f, g$ are measurable then: $\|fg\|_1 \le \|f\|_p \, \|g\|_q$. In particular for $f \in L^p$ and $g \in L^q$ then $fg \in L^1$ and equality holds if and only if $f^p = kg^q$ almost everywhere for some $k$.

*Proof.* If $\|f\|_p^p = 0 \implies f = 0$ a.e., hence $\|f\|_p^p$ or $\|g\|_p^p = 0 \implies fg = 0$ a.e. and inequality holds. Similarly if $\|f\|_p$ or $\|g\|_q$ is $\infty$ inequality also holds. We ca now assume that $0 < \|f\|_p, \|g\|_q \, \infty$. We have a standard inequality $a, b \ge 0$, $0 < \lambda < 1$: $a^\lambda b^{1-\lambda} \le \lambda a + (1-\lambda)b$ with equality only if $a = b$. Let $a := \frac{|f(x)|^p}{\|f\|_p^p}$ and $b := \frac{|g(x)|^q}{\|g\|_q^q}$ then for $\lambda = \frac{1}{p}$ we have:

$$a^{\frac{1}{p}} b^{\frac{1}{q}} = \frac{|f(x)|}{\|f\|_p} \frac{|g(x)|}{\|g\|_q} \le \frac{1}{p} \frac{|f(x)|^p}{\|f\|_p^p} + \frac{1}{q} \frac{|f(x)|^q}{\|f\|_p^q}$$

then:

$$\frac{1}{\|f\|_p^p \|g\|_q^q} \int |fg| \, d\mu \le \frac{1}{p \|f\|_p^p} \int |f|^p \, d\mu + \frac{1}{q \|g\|_q^q} \int |g|^q \, d\mu = \frac{1}{p} + \frac{1}{q} = 1$$

hence:

$$\|fg\|_1 \le \|f\|_p \, \|g\|_q$$

This is only equal if $\frac{|f|^p}{\|f\|_p^p} = \frac{|g|^q}{\|g\|_q^q} \iff |f|^p = |g|^q$ a.e.. □

**Corollary 2.4.6: Minkowski Inequality**

For $p \ge 1$ then $\|-\|_p$ is a norm.

*Proof.* Note that for $p = 1$ we have nothing to prove, so suppose $p > 1$. Then we have $(p-1)q = p$ and $f, g \in L^p \implies f + g \in L^p \implies |f + g|^{p-1} \in L^q$ and so $\left(\int |f + g|^{p-1}\right)^q = \int |f + g|^p < \infty$. Then:

$$|f + g|^p \le (|f| + |g|) \, |f + g|^{p-1} \qquad |f + g|^{p-1} \in L^q$$

so:

$$\begin{aligned}\int |f+g|^p \, d\mu &\leq \int (|f|+|g|)\,|f+g|^{p-1}\, d\mu \\ &\leq \|f\|_p \left\|(f+g)^{p-1}\right\|_q + \|g\|_p \left\|(f+g)^{p-1}\right\|_q \qquad \text{By Hölder} \\ &\leq (\|f\|_p + \|g\|_q) \left\|(f+q)^{p-1}\right\|_q \\ &\leq (\|f\|_q + \|g\|_p)(\|f+g\|_p^p)^{\frac{1}{q}} \\ &= (\|f\|_p + \|g\|_p)\, \|f+g\|^{\frac{p-1}{p}}\end{aligned}$$

so:

$$\|f+g\|_p \leq \dots$$

□

Also $\|f\|_p = 0 \implies \int |f|^p \, d\mu = 0 \implies f = 0$ a.e. so $\|-\|_p$ is a norm on $L^p$.

**Theorem 2.4.7**

For $p \geq 1$ we have $L^p$ Banach.

*Proof.* We will prove that all absolute convergent series are convergent series. Let $\Sigma \|f_n\| < \infty$. Define $F_N = \sum_1^N |f_n|$, then $\forall N : \|F_N\|_p = \sum_1^N \|f_n\|_p \leq \sum_1^\infty \|f_n\|_P = K < \infty$.
So we have: $\forall N : \int f_N^p d\mu \leq k^p$, then by Monotone convergence of $|F_N|^p$ we have that $F := \sum_1^\infty |f_n|$, $F_N \uparrow F$ so:

$$\int F_n^p \to \int F^p \implies \int F^p \leq K^p$$

hence $F \in L^p$ and $F < \infty$ a.e. so we have the $G(x) = \sum_1^\infty f_n(x)$ is absolutely convergent a.e. and since $|G| \leq F_n \implies G \in L^p$. We now want to show $\sum_1^N f_n \to G$ in $L^p$:

$$\left|G - \sum_1^n f_n\right| \leq 2^p F^p$$

then by LCT we have that:

$$\lim_{N\to\infty} \int \left|G - \sum_1^N f_n\right|^p d\mu \to \int 0 = 0 \implies \lim_{N\to\infty} \left\|G - \sum_1^N f_n\right\|_p^p \to 0 \implies \sum_1^N f_n \xrightarrow{L^p} G$$

□

**Proposition 2.4.8**

For $p \geq 1$, simple functions: $\{\sum a_j \chi_{E_j} | \mu(E_j) < \infty\}$ are dense.

*Proof.* Given $f \in L^p$. Let $\{f_n\}$ simple $|f_n| \leq |f|$ $f_n \to f$, then $f_n \in L^+$ and $|f_n - f|^p$ '$2|f|^p$ is integrable, then by LDC:

$$\lim_{n\to\infty} \int |f_n - f|^p \, d\mu \to \int 0 = 0$$

So $\|f_n - f\|_p \to 0$ i.e. $f_n \xrightarrow{f_p} f$. Note we can assume: $f_n = \sum a_j \chi_{E_j}$ with $\mu(E_j) > 0$, disjoint:

$$\|f_n\| - \sum_{j=1}^n |a_j|^n \mu(E_j) \infty \implies \mu(E_j) < \infty$$

□

### 2.4.1 $L^\infty$

We have a norm $\|f\|_\infty := \{\inf k \geq 0 | \mu\{x| |f(x)| > k\} = 0\}$ which is called the essential supremum. If $\|f\|_\infty < \infty \exists M$ such that $|f| \leq M$ a.e.. Note that the inf is attained since:

$$\{x|f(x) > a\} = \cup_1^\infty \{x|f(x) > +\frac{1}{n}\}$$

**Definition 2.4.9**

We define the space $L^\infty := \{f : X \to \mathbb{C} | \text{measurable}, \|f\|_\infty < \infty\}$.

**Theorem 2.4.10**

- If $f, g$ measurable then: $\|fg\|_1 \leq \|f\|_1 \|g\|_\infty$ and if $f \in L^1, g \in L\infty$ equality holds if and only if $g = \|g\|_\infty$ a.e. on $\{x|f(x) \neq 0\}$.
- $\|-\|_\infty$ is a norm.
- $\|f_n - f\|_\infty \to 0 \iff \exists E \in \mathcal{M} : \mu(E^c) = 0$ and $f_n \to f$ uniformly on $E$.
- $L^\infty$ is a Banach space.
- simple functions are dense.

### 2.4.2 Duals of $L^p$

We have that $L^2$ is an Hilbert space hence $(L^2)^* \cong L^2$ via $g \mapsto \phi_g$ where $\phi_g(f) = \int f\overline{g}$. In general we have $(L^p)^* = L^q$ for $p \neq 1$. We have that for $g \in L^q$: $\phi_g \in L^q, \phi_g(f) := \int fg$ which is in $(L^p)^*$ since:

$$|\phi_g(f)| = \left|\int fg\right| \leq \int |fg| \leq \|f\|_p \|g\|_q \quad \text{Hölder}$$

We then have that $\|\phi_g\| \leq \|g\|_q$, then:

**Theorem 2.4.11**

If $1 \leq q < \infty$ then $\phi_g \in (L^p)^*$ and $|\phi_g| = \|g\|_q$( if $\mu$ is semi-finite is true for $q = \infty$).

*Proof.* We need to find $f \in L^n$ $\|f\|_p = 1$ and $|\phi_g(f)| = \|g\|_q$ take $f := \frac{g^{q-1}\overline{\text{sgn}(g)}}{\|g\|_q^{q-1}} \in L^p$ and also:

$$\|f\|_p^p = \frac{1}{\|g\|_q^{p(q-1)}} \left(\int |g|^{p(q-1)}\right) = \frac{\|g\|_q^{\frac{q}{p}}}{\|g\|_q^{p(q-1)}} = 1$$

since $p(q-1) = \frac{q}{p}$. Then $\phi_g(f) = \int \frac{|g|^q}{\|g\|_q^{q-1}} = \|g\|_q$, hence $g \mapsto \phi_g$ is an isometry to its image.
We have $L^q \to (L^p)^*$ and $g \to \phi_g$ is an isometry for $1 \leq q \leq \infty$ (for $\mu$ semi-finite). □

Define $\Sigma = \{f : X \to \mathbb{C} | \text{meas. simple that vanish outside some } E, \mu(E) < \infty\}$, then we have that $\forall p : \Sigma \subseteq L^p$.

**Theorem 2.4.12**

Let $g$ measurable and $fg \in L^1 \forall f \in \Sigma$ and $\hat{\phi}_g : \Sigma \to \mathbb{C}$, $f \mapsto \int fg$ is bounded with respect to $L^p$ norm, i.e. $|\hat{\phi}_g(f)| \leq \|\hat{\phi}_g\| \, \|f\|_p$. Then for $S_g = \{x | g(x) \neq 0\}$ and $\sigma$-finite or $\mu$-semi-finite, then $g \in L^q$ and $\|\hat{\phi}_g\| = \|g\|_q$.

*Proof.* For $f$ bounded measurable and vanishing outside $E, \mu(E) < \infty$, $|f(x)| < M$. We then have $f_n \uparrow |f|$. By assumption $\chi_E \in \Sigma$ then $\chi_E g \in L^1$. Also: $|f_n| \leq M |\chi_E g| \in L^1$, then by LDC: $\int f_n g \to \int fg$ and $|\int fg| = \lim_{n\to\infty} |\int f_n g| \leq |\hat{\phi}_g| \, \|f_n\|_p \dots$.
Let $S_g := \cup_1^\infty E_n$, $\mu(E_n) < \infty$ and $\sigma$-finite, $\phi_n$ simple and $|\phi_n| \uparrow |g|$ and $g = \lim_{n\to\infty} \phi_n \chi_{E_n}$ and $g_n := \phi_n \chi_{E_n}$, for $f_n = \frac{|g_n|^{q-1} \text{sgn}(g)}{\|g_n\|_q^{q-1}}$ simple, vanishes outside finite area. Then:

$$\begin{aligned}
\|g\|_q &= \left( \int |g|^q \right)^{\frac{1}{q}} = \left( \int \lim_{n\to\infty} |g_n|^q \right)^{\frac{1}{q}} = \\
&\leq \liminf \left( \int |g_n|^q \right)^{\frac{1}{q}} = \liminf \left( \int |f_n g_n|^q \right)^{\frac{1}{q}} = \\
&\leq \|\hat{\phi}_g\|
\end{aligned}$$

hence: $\|g\|_q < \infty \implies g \in L^q$ and $\|g\| = \|\hat{\phi}_g\|$ (by Hölder). □

**Theorem 2.4.13**

The map $\phi : L^q \to (L^p)^*$ is a linear isometry for $1 < p < \infty$ ( and $p = 1$ for $\mu$ $\sigma$-finite).

*Proof.* If $\mu$ is finite $\implies L^p$ contains all simple $\phi \in (L^p)^*$, define: $\nu(E) = \phi(\chi_E)$, this gives a complex measure and $\mu(E) = 0 \implies \chi_E = 0$ a.e., hence $\nu(E) = \phi(\chi_E) = 0 \implies \nu \ll \mu$. So $\exists g \in L^1$ such that $\forall E : \nu(E) = \phi(\chi_E) = f g \chi_E d\mu \implies \phi = \phi_g$ on simple functions $\implies \phi = \hat{\phi}_g$ for $g \in L^q$. □

## 2.5 Dynamics

Say there exists a map $T$ such that $\forall E \in \mathcal{M} : \mu(T^{-1}E) = \mu(E)$.

**Example.** Consider $f(z) = z^2$, then $f : S^1 \to S^1$ with the Lebesgue measure. For $E \in S^1$: $\mu(f^{-1}(E)) = \mu(E)$ since it will just be 2 copies with half the measure of $E$. Alternatively $f : [0,1] \to [0,1]$ given by $x \mapsto [2x]_1$.

**Definition 2.5.1**

A dynamical system is $(X, \mathcal{M}, \mu, T)$ with $T : X \to X$ measurable and $\mu(T^{-1}E) = \mu(E)$, i.e. it preserves $\mu$. Note it is a bijection.

**Theorem 2.5.2: Poincaré Recurrence**

Let $(X, \mathcal{M}, \mu, T)$ be a finite dynamical system ($\mu(X) < \infty$). Then for any $E \in \mathcal{M}$ the orbit of a point $x$ return to $E$ infinitely many times for almost every $x \in E$.

*Proof.* Let $A_n := \cup_{k=n}^{\infty} T^{-k}(E)$, for $T^{-k}(E) = \{x \in X | T^k(x) = E\}$, so $A_n$ is the set of points mapping to $E$ after at least $n$ iterations. Now note that $E \subset A_0$ and then $A_{i+1} \subset A_i$, moreover $T^{-1}(A_i) = A_{i+1}$ and since $T$ is $\mu$ invariant we get that $\forall i, j : \mu(A_i) = \mu(A_j)$. Now we have that:

$$\forall n : 0 \leq \mu(E \setminus A_n) \leq \mu(A_0) - \mu(A_n) = 0$$

So the set of points in $E$ returning after $n$ or more iterations is full measure.
Then by looking at: $\cap_{i=1}^{\infty} A_n$ which is the set of infinite returns we get that:

$$\mu(E \setminus \cap_{n=1}^{\infty} A_n) = \mu(\cup_{i=1}^{\infty} E \setminus A_n) \leq \sum_{n=1}^{\infty} \mu(E \setminus A_n) = 0$$

and we are done. □

**Theorem 2.5.3: Birkoff's Ergodic Theorem**

Let $f \in L^1(X, \mathbb{R})$ and $(X, \mathcal{M}, \mu, T)$ be a finite dynamical system. then there exists some $\overline{f} \in L^1(X, \mathbb{R})$ such that:

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(x)) = \overline{f}(x) \qquad \mu - a.e.$$

so the limit of the average of the orbit of iteration is in $L^1$.

*Proof.* Let $A_n(f)(x) := \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(x))$. Consider $f = \chi_E$ for $E \in \mathcal{M}$, then we have that: $A_n(f)(x) = \frac{|\{k: 0 \leq k \leq n-1 | T^{-k}(x) \in E\}|}{n}$.
Clearly $A_n(f) \in L^+$ since is a sum of $L^+$ functions, moreover by $\mu$-invariance of $T$ we have: $\forall n : \int A_n f(x) d\mu = \int f d\mu$.
Let $\overline{A}$ and $\underline{A}$ be the $\limsup$ and $\liminf$ respectively of $A_n f$. We will show that $\int \overline{A} d\mu \leq \mu(E) \leq \int \underline{A} d\mu$, since then:

$$0 \leq \int \overline{A} - \underline{A} d\mu \leq 0$$

and so $\overline{A} = \underline{A}$ $\mu$-a.e..
Fix $\epsilon > 0$ and define $\tau(x) = \min\{n | A_n f(x) \leq \overline{A} - \epsilon\}$. We have two cases:

1. there is $M_\epsilon > 0$ such that $\forall x : \tau(x) \leq M$, then $A_n f(x) \geq \frac{n-m}{n}(\overline{A}(x) - \epsilon)$ by partitioning the orbit in sets of size $m$ and the using the upper bound. Then $\forall n$ :

   $$\mu(E) = \int A_n f(x) d\mu \geq \frac{n-m}{n} \int \overline{A} - \epsilon d\mu$$

   so by letting got to infinity we get:

   $$\mu(E) \geq \int \overline{A} d\mu - \epsilon \mu(x)$$

   which then by letting $\epsilon$ go to zero we are done.

2. If $\tau(x)$ is not bounded, let $B_k := \{x \in E | \tau(x) \leq k\}$ so $X = \cup_k B_k$ and $B_k \subset B_{k+1}$. Hence, there is $N$ such that $\mu(B_N^c \cap E) \leq \epsilon$.
   Now let $E' := E \cup B_N^C$ and define $S_n' := \frac{|\{k \in \{0, \ldots, n-1\} | T^k(x) \in E\}|}{n}$, as before $S_n'(x) \geq (n-N)(\overline{A}(x) - \epsilon)$, by considering the partition, so $\mu(E') \geq \int (\overline{A}(x) - \epsilon) dmu = \int \overline{A}(x) d\mu - \epsilon \mu(X)$ which implies that:

   $$\int \overline{A}(x) d\mu - \epsilon \mu(X) \leq \mu(E) + \epsilon$$

   then by letting $\epsilon \to 0$ we get:

   $$\int \overline{A} d\mu \leq \mu(E) = \int f d\mu$$

   then by doing everything for $\underline{A}$ we get the other inequality, and so our claim holds.

We just proved the result for $f = \chi_E$, then it holds by linearity for simple function and so by Monotone Convergence we can extend it to $L^+$ and so to $L^1$ by linearity. □

**Remark.** We also have that $\int \overline{f} = \int f$.

**Definition 2.5.4**

A function $T : X \to X$ $\mu$-invariant is said ergodic if for all $E \in \mathcal{M}$ such that $T^{-1}E = E$ then either $\mu(E) = 0$ or $\mu(E^C) = 0$.

**Example.** Rotations by an irrational angle on $S^1$ are ergodic while ration rotations are not since $\epsilon$-neighbourhoods of orbits are fixed as well.

Alternatively $T : X \to X$ is ergodic if $\forall f \in L^1(X)$ such that $f \circ T = f$ a.e. implies that $f = c$ a.e. for some constant.

**Corollary 2.5.5**

If $T$ is ergodic then: $\lim_{n\to\infty} f1n \sum_{k=0}^{n-1} f(T^k(x)) = \frac{\int f d\mu}{\int d\mu}$.

*Proof.* We have that: $\overline{f}(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(x))$ a.e. then $\overline{f} \circ T = \overline{f} \implies \overline{f} = c$ a.e. so:

$$\int \overline{f} d\mu = \int f d\mu \implies c\mu(X) = \int f d\mu$$

□

So $T$ ergodic implies that the Time average is a Space average.

**Definition 2.5.6**

We say $(X, \mathcal{M}, \mu)$ is a probability measure space if $\mu(X) = 1$[a].

[a] Note that if $(X, \mathcal{M}, \mu)$ is finite measure then we can put a new measure $\tilde{\mu}(E) := \frac{\mu(E)}{\mu(X)}$ which then makes $(X, \mathcal{M}, \tilde{\mu})$ a probability space.

**Definition 2.5.7**

For $x \in \mathbb{R}$ we say it is base 2 normal if for all $x = x_1 x_2 x_3 \cdots$ in binary then:

$$\lim_{n\to\infty} \frac{\sharp\{x_i = 1 | i \le n\}}{n} = \frac{1}{2}$$

**Theorem 2.5.8**

Let $E \subset \mathbb{R}$ be the set of base 2 normal point, then: $\mathfrak{m}(E^C) = 0$; i.e. almost every point is normal.

*Proof.* Consider $T : [0,1) \to [0,1)$ given by $T(x) = 2x \mod 1$, then $T(.x_1x_2\ldots) = .x_2x_3\ldots$, then $f$ is $\mathfrak{m}$-invariant and ergodic (it pulls-back interval to 2 intervals with half size).
Let $E = T^{-1}(E)$ and assume $\mathfrak{m}(E) > 0$,, let $x$ be a Lebesgue density point, $\lim_{r\to 0} \frac{\mathfrak{m}(B(x,r)\cap E)}{\mathfrak{m}(B(x,r))} = 1$. For $B_r \subset B(x,r)$ shrinking nicely, we have the same: $\lim_{r\to 0} \frac{\mathfrak{m}(B_r \cap E)}{\mathfrak{m}(B_r)}$.

Since: $\frac{n_k}{2^k} \leq x \leq \frac{n_{k+1}}{2^{k+1}}$, let $I_k = [\frac{n_k}{2^k}, \frac{n_{k+1}}{2^{k+1}}]$, then for each $k$:

$$\lim_{k\to\infty} \frac{\mathfrak{m}(I_k \cap E)}{\mathfrak{m}(I_k)} = 1$$

but: $T^{-1}(I_k \cap E) = T^{-1}(I_k) \cap E$ which implies $T^{-k}(I_k \cap E) = T^{-k}(I_k) \cap E$ and $\forall k : T^{-k}(I_k) = [0,1)$, thus:

$$\forall k : \mathfrak{m}(T^{-k}(I_k \cap E)) = \mathfrak{m}(T^{-k}(I_k) \cap E) = \mathfrak{m}([0,1) \cap E) = \mathfrak{m}(E) > 0$$

but then we have a contradiction since the limit: $\lim_{k\to\infty} \frac{\mathfrak{m}(E)}{\mathfrak{m}(I_k)} = 1$ $\sharp$.
Now to prove that normal points are full measure, let: $T(x) = 2x \mod 1$ on $[0,1)$ and let $f(x) = \chi_{[0,\frac{1}{2})}$, then:

$$\frac{1}{n}\sum_{0}^{n-1} f(T^k(x)) = \frac{|\{i \leq m | x_i = 0\}|}{n}$$

the relative frequency of 0's in the $n$-th binary expansion. By the Ergodic theorem we have:

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} f(T^k(x)) = \int f(x) d\mathfrak{m} = \frac{1}{2} \quad \mathfrak{m}\text{-a.e.}$$

□

If we look at continuous fractions expressions: $x = [a_0 a_1 \ldots a_n \ldots]$ and consider $T : [0,1) \to [0,1)$ given by $T(x) = \left\lceil \frac{1}{x} \right\rceil$ and $T^k(x) = a_k$.