MATH 541a Homework 6

Qilin Ye

April 14, 2022

Problem 1

For random variables X, Y, define

$$\operatorname{var}(X \mid Y) := \mathbb{E}[(X - \mathbb{E}(X \mid Y))^2 \mid Y].$$

In this exercise you can freeuse the identity

$$\operatorname{var}(X) = \mathbb{E}[\operatorname{var}(X \mid Y)] + \operatorname{var}[\mathbb{E}(X \mid Y)].$$

Give a different proof of the Rao-Blackwell Theorem when the loss function is the mean squared error.

Proof. Let Z be sufficient for f_θ and let Y be unbiased for $g(\theta)$. Define $W := \mathbb{E}_\theta(Y \mid Z)$. Then we have

$$\begin{aligned}
\operatorname{var}(W) &= \mathbb{E}(W - g(\theta))^2 \\
&= \mathbb{E}[\mathbb{E}(Y \mid Z) - g(\theta)]^2 \\
&= \mathbb{E}[\mathbb{E}(Y \mid Z) - \mathbb{E}(g(\theta) \mid Z)]^2 \\
&= \mathbb{E}[\mathbb{E}(Y - g(\theta) \mid Z)]^2 \\
&\leqslant \mathbb{E}[\mathbb{E}(Y - g(\theta))^2 \mid Z] \\
&= \mathbb{E}(Y - g(\theta))^2 = \operatorname{var}(Y).
\end{aligned}$$

□

Problem 2

Let $X_1, \ldots, X_n$ be a random sample of size n so that X_1 is a sample from the uniform distribution on $[\theta - 1/2, \theta + 1/2]$ where $\theta \in \mathbb{R}$ is unknown.

(1) Let $U = u(X_1)$ be an unbiased estimator of 0 where $u : \mathbb{R} \to \mathbb{R}$. By differentiating the definition of unbiasedness w.r.t. θ conclude that

$$u(x + 1) = u(x) \qquad \text{for a.e. } x \in \mathbb{R}.$$

Give an example of an unbiased estimator U of 0 such that $u(x) \neq 0$ for all $x \in \mathbb{R}$.

(2) Assume that W is UMVU for $g(\theta)$. Using the characterization from class, conclude that $\mathbb{E}_\theta WU = 0$ so

that if $W = w(X_1)$ with $w : \mathbb{R} \to \mathbb{R}$, then

$$w(x+1)u(x+1) = w(x)u(x) \qquad \text{for a.e. } x \in \mathbb{R}.$$

Then conclude that

$$w(x+1) = w(x) \qquad \text{for a.e. } x \in \mathbb{R}.$$

(3) To complete the proof that UMVU does no exist, what can you saya bout the condition that W is unbiased for $g(\theta)$?

Proof. In this case the expected value is simply $\int_{\theta-1/2}^{\theta+1/2} u(x)\,\mathrm{d}x = 0$. Hence $u(\theta+1/2) = u(\theta-1/2)$ for a.e. $\theta \in \mathbb{R}$. For the unbiased mean zero estimator that is nowhere zero, consider

$$u(x) := \begin{cases} \operatorname{sgn}(x)/x^2 & |x| \geqslant 1 \\ \operatorname{sgn}(x) & 0 < |x| < 1 \\ 1 & x = 1. \end{cases} \tag{*}$$

For the UMVU part, if W is UMVU, then since U is unbiased for 0, we must have $\mathbb{E}_\theta WU = 0$ by the characterization. Using a similar argument as above we have

$$w(x+1)u(x+1) = w(x)u(x) \qquad \text{for a.e. } x \in \mathbb{R}. \tag{**}$$

Using the U defined in (*), we may safely divide both sides of (**) by u and obtain $w(x+1) = w(x)$ for a.e. $x \in \mathbb{R}$. This, together with the fact that W is unbiased, implies $g(\theta)$ must be constant, since

$$\int_{\theta-1/2}^{\theta+1/2} w(x)\,\mathrm{d}x = 0 \implies g'(\theta) = w(\theta+1/2) - w(\theta-1/2) = 0.$$

□