Beginning of Jan.12, 2022

Expected Value and Variance

Notation: given $A \subset \Omega$, we define the **indicator function** $1_A : \Omega \to \{0,1\}$ by

$$1_A(\omega) := \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A. \end{cases}$$

Definition 0.0.1: (1.37) Expected Values

Let $\mathbb{P}$ be a probability law on Ω and let $X : \Omega \to [0,\infty]$. Define the **expected value** of X denoted $\mathbb{E}X$ to be

$$\mathbb{E}X := \int_0^\infty \mathbb{P}(X > t)\, \mathrm{d}t.$$

A simple application of Tonelli shows that if X is continuous then $\mathbb{E}X$ agrees with $\int_{-\infty}^{\infty} x f_X(x)\, \mathrm{d}x$ which we are more familiar with. If X is discrete, the analogous version is $\mathbb{E}X = \sum_{k\in\mathbb{R}} k\mathbb{P}(X = k)$.
In particular, if $X : \mathbb{R} \to \mathbb{R}$ and if $\mathbb{E}|X| < \infty$, then we can define

$$\mathbb{E}X := \mathbb{E}X^+ - \mathbb{E}X^-$$

where

$$X^+ := \max\{X, 0\} \qquad \text{and } X^- := \max\{-X, 0\}.$$

Remark. If $X : \Omega \to [0,\infty)$, then for positive integer n,

$$\mathbb{E}X^n = \int_0^\infty n t^{n-1}\mathbb{P}(X > t)\, \mathrm{d}t.$$

More generally, if $g : [0,\infty) \to [0,\infty)$ continuous differentiable with $g(0) = 0$, then

$$\mathbb{E}g(X) = \int_0^\infty g'(t)\mathbb{P}(X > t)\, \mathrm{d}t.$$

Proposition: (1.43) Linearity of $\mathbb{E}$

Let $X_1, \ldots, X_n$ be random variables. Then $\mathbb{E}(\sum_{i=1}^n X_i) = \sum_{i=1}^n \mathbb{E}X_i$.

Definition: (1.44) Variance

If $\mathbb{E}|X| < \infty$, define $\mathrm{var}(X) := \mathbb{E}(X - \mathbb{E}X)^2 = \mathbb{E}X^2 - (\mathbb{E}X)^2$ to be the **variance** of X.

Remark. If $X : \Omega \to \mathbb{C}$ is complex valued, then if $\mathbb{E}|X| < \infty$, we can define

$$\mathbb{E}X := \mathbb{E}\mathfrak{Re}(X) + i\mathbb{E}\mathfrak{Im}(X)$$

and $\mathrm{var}(X) := \mathbb{E}(X - \mathbb{E}X)^2$ as before.

Joint Distributions

Definition: (1.47) Joint PDF

A **joint PDF** for two random variables is a function $f : \mathbb{R}^2 \to [0, \infty)$ with

$$\iint_{\mathbb{R}^2} f(x, y) \, \mathrm{d}x\mathrm{d}y = 1$$

and such that

$$\int_c^d \int_a^b f_{X,Y}(X, Y) \, \mathrm{d}x\mathrm{d}y$$

exists for all $[a, b] \times [c, d] \in \overline{\mathbb{R}}^2$.

We say X, Y are **jointly continuous** with joint PDF $f_{X,Y}$ if

$$\mathbb{P}((X, Y) \in A) = \iint_A f_{X,Y}(x, y) \, \mathrm{d}x\mathrm{d}y \qquad \text{for “all” } A \subset \mathbb{R}^2.$$

Definition: (1.48) Marginals

We define the **marginal PDF** f_X of X to be

$$f_X(x) := \int_{-\infty}^{\infty} f_{X,Y}(x, y) \, \mathrm{d}y \qquad \text{for all } x \in \mathbb{R}.$$

Similarly, if $g : \mathbb{R}^2 \to \mathbb{R}$, we define

$$\mathbb{E}g(X, Y) := \iint_{\mathbb{R}^2} g(x, y) f_{X,Y}(x, y) \, \mathrm{d}x\mathrm{d}y.$$

Definition: (1.55) Independence of RVs

Let $X_1, \dots, X_n$ be r random variables on Ω. We say they are **independent** if

$$\mathbb{P}(X_1 \leqslant x_1, \dots, X_n \leqslant x_n) = \prod_{i=1}^n \mathbb{P}(X_i \leqslant x_i) \qquad \text{for all } (x_1, \dots, x_n) \in \mathbb{R}^n.$$

In particular if $X_1, \dots, X_n$ are continuous, then the definition is equivalent to saying

$$f_{X_1,\dots,X_n}(x_1, \dots, x_n) = \prod_{i=1}^n f_{X_i}(x_i) \qquad \text{for all } (x_1, \dots, x_n) \in \mathbb{R}^n.$$

Proposition: (1.59, 1.60)

If $X_1, \dots, X_n$ are independent and $\mathbb{E}X_i < \infty$, then

$$\mathrm{var}\Big(\sum_{i=1}^n X_i\Big) = \sum_{i=1}^n \mathrm{var}(X_i),$$

and

$$\mathbb{E}\Big(\prod_{i=1}^n X_i\Big) = \prod_{i=1}^n \mathbb{E}(X_i).$$

Conditional Probability

Let $A, B \subset \Omega$ with $\mathbb{P}(B) > 0$. We define

$$\mathbb{P}(A \mid B) := \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

and read the **probability of A given B.**

For a fixed B, we define

$$\mathbb{E}(X \mid B) := \frac{\mathbb{E}X \cdot 1_B}{\mathbb{P}(B)}.$$

Proposition: Laws of Total Probability & Expectation

If $A \subset \Omega$ and $\{B_i\}$ partitions Ω, then

$$\mathbb{P}(A) = \sum_{i=1}^{\infty} \mathbb{P}(A \cap B_i) = \sum_{i=1}^{\infty} \mathbb{P}(A \mid B_i)\mathbb{P}(B_i)$$

and

$$\mathbb{E}X = \sum_{i=1}^{\infty} \mathbb{E}(X 1_{B_i}) = \sum_{i=1}^{\infty} \mathbb{E}(X \mid B_i)\mathbb{P}(B_i).$$

Definition: (1.75) Conditioning a RV

Let X, Y be continuous random variables with joint PDF $f_{X,Y}$. Fix $y \in \mathbb{R}$ with $f_Y(y) > 0$. Then for any $x \in \mathbb{R}$ we define the **conditional PDF** of X given $Y = y$ by

$$f_{X|Y}(x \mid y) := \frac{f_{X,Y}(x, y)}{f_Y(y)}.$$

The **conditional expectation** is given by

$$\mathbb{E}(X \mid Y = y) = \int_{-\infty}^{\infty} x f_{X|Y}(x \mid y) \, \mathrm{d}x.$$