1 Semigroups

Definition 1.1: Semigroup

We say $T = \{T(t)\}_{t \geqslant 0}$ is a **semigroup** of bounded linear operators on a Banach X if:

(1) $T(0) = \text{id}$, and

(2) $T(t+s) = T(t)T(s)$ for all $t, s \geqslant 0$.

We introduce two types of semigroups —

(1) T is **uniformly continuous** if $\|T(t) - \text{id}\| \to 0$ (in operator norm) as $t \to 0$. In particular, $\|T(s) - T(t)\| \to 0$ whenever $t \to s$, since

$$\|T(s) - T(t)\| = \|T(s)\text{id} - T(s)T(t-s)\| \leqslant \|T(s)\| \underbrace{\|\text{id} - T(t-s)\|}_{\to 0}.$$

(2) **Strongly continuous**; see the next page.

Uniformly Continuous Semigroups

Definition 1.2: Generator

We define the linear operator

$$Ax := \lim_{t \to 0} \frac{T(t)x - x}{t}$$

to be the **infinitesimal generator** of the $T(t)$. Its domain is

$$D(A) := \left\{ x \in X : \lim_{t \to 0} \frac{T(t)x - x}{t} \text{ exists} \right\}.$$

Theorem 1.3

A is the infinitesimal generator of some uniformly continuous $T(t)$ if and only if A is bounded (with characterization $T(t) = e^{At} := \sum_{n \geqslant 0} (tA)^n / n!$).

Corollary 1.4

If A is an infinitesimal generator of a uniformly continuous $T(t)$ then $D(A) = X$.

Theorem 1.5

Infinitesimal generators of uniformly continuous semigroups are unique.

Corollary 1.6

Let $T(t)$ be a uniformly continuous semigroup. Then:

(1) There exists a *unique* a bounded linear operator $A \in B(X)$ such that $T(t) = e^{At}$ where A is the infinitesimal generator.

(2) There exists $c \geqslant 0$ such that $\|T(t)\| \leqslant e^{ct}$.

(3) The mapping $t \mapsto T(t)$ is differentiable in norm, i.e., for all t_0, there exists an bounded operator $\frac{\mathrm{d}}{\mathrm{d}t}T(t_0)$ such that

$$\lim_{h \to 0} \left\| \frac{T(t_0 + h) - T(t_0)}{h} - \frac{\mathrm{d}}{\mathrm{d}t}T(t_0) \right\|_{B(X)} = 0.$$

Furthermore, $\frac{\mathrm{d}}{\mathrm{d}t}T(t) = AT(t) = T(t)A$.

Strongly Continuous Semigroups

Definition 1.7: Strongly Continuous Semigroups

Let $T(t)$ be a semigroup of bounded operators. We say T is **strongly continuous** if

$$\lim_{t \to 0} T(t)x = x \qquad \text{for all } x \in X.$$

Then T is called a $\boldsymbol{C_0}$ **semigroup**. *Note this is only a pointwise property albeit the name "strongly".*

Theorem 1.8

Let T be a C_0 semigroup. Then:

(1) There exists $w \geqslant 0, n \geqslant 1$ such that $\|T(t)\| \leqslant Me^{\omega t}$.

(2) For any fixed x, $T(t)x \in C([0, \infty); x)$.

Proof. First note that $\|T(t)\|$ is bounded on $[0, \eta]$ for some $\eta > 0$. (Otherwise, there exists a sequence $t_n \searrow 0$ such that $\|T(t_n)\| \to \infty$, whereas the definition gives $T(t)x \to x$ and in particular $\|T(t_n)x\| \to \|x\|$, i.e., $\|T(t_n)x\|$ is bounded pointwise. This contradicts the uniform boundedness principle.)

Therefore there exists $M \geqslant 1$ such that $\|T(t)\| \leqslant M$ for all $t \in [0, \eta]$. Now we set $\omega := \eta^{-1} \log M$. Any $t \geqslant 0$ can be represented as $t = n\eta + \delta$ where $n \in \mathbb{N}$ and $\delta \in [0, \eta)$. An iterative argument shows that

$$\|T(t)\| = \|T(\eta)^n T(\delta)\| \leqslant= MM^n < MM^{t/\eta} = M(e^{\omega})^t = Me^{\omega t}.$$

Now we show (2). We will show that $T(t)x$ is continuous on $C([0, t_0); x)$ for any t_0. Let t_0 be given and fix $\epsilon > 0$.

Let $\delta > 0$ be such that $\|T(t)x - x\| \leqslant \epsilon$ whenever $t \leqslant \delta$. Then for $s \leqslant t \leqslant t_0$,

$$\begin{aligned}\|T(t)x - T(s)x\| &= \|T(s)T(t-s)x - T(s)x\| \\ &\leqslant \|T(s)\|\|T(t-s)x - x\| \\ &\leqslant Me^{\omega s}\|T(t-s)x - x\| \leqslant Me^{\omega t_0}\epsilon \qquad \text{(from part (1))}\end{aligned}$$

if $t - s \leqslant \delta$. Since ϵ is arbitrary we are done. □

Theorem 1.9

Let T be a C_0 semigroup and let A be its infinitesimal generator. Then:

(1) $\lim_{h\to 0} \frac{1}{h}\int_t^{t+h} T(s)x \,\mathrm{d}s = T(t)x$ for all $x \in X$ and $t \geqslant 0$.

(2) $\int_0^t T(s)x \,\mathrm{d}s \in D(A)$ for all $x \in X$. Moreover, $A\left(\int_0^t T(s)x \,\mathrm{d}s\right) = T(t)s - x$.

(3) If $x \in D(A)$ then $T(t)x \in D(A)$ and $\frac{\mathrm{d}}{\mathrm{d}t}T(t)x = AT(t)x = T(t)Ax$.

(4) For $x \in D(A)$, we have $T(t)x - T(s)x = \int_s^t T(\tau)Ax \,\mathrm{d}\tau = \int_s^t AT(\tau)x \,\mathrm{d}\tau$.

(*Compare some of these with the fundamental theorem of calculus. Also, if $A, B \in L(X)$ are unbounded, it might happen that $ABx \neq BAx$ for some x. Even in finite-dimensional settings (e.g. matrices).*)

Proof. (1)

$$\begin{aligned}\left\|\frac{1}{h}\int_t^{t+h} T(s)x \,\mathrm{d}s - T(t)x\right\| &= \left\|\frac{1}{h}\int_t^{t+h} T(t)T(s-t)x \,\mathrm{d}s - \frac{1}{h}\int_t^{t+h} T(t)x \,\mathrm{d}s\right\| \\ &\leqslant \|T(t)\| \cdot \frac{1}{h}\int_t^{t+h} \|T(s-t)x - x\| \,\mathrm{d}s.\end{aligned}$$

The norm in the integrand can be made sufficiently small if $s - t$ is sufficiently small. Therefore everything can be made sufficiently small.

(2) To show this claim, we need to prove that $\lim_{h\to 0} \frac{T(h) - \mathrm{id}}{h}\int_0^x T(s)s \,\mathrm{d}s$ exists:

$$\begin{aligned}\frac{T(h) - \mathrm{id}}{h}\int_0^x T(s)x \,\mathrm{d}x &= \frac{1}{h}\int_t^{t+h} T(s)x \,\mathrm{d}s - \frac{1}{h}\int_0^x T(s)x \,\mathrm{d}x \\ &= \frac{1}{h}\int_t^{t+h} T(s)s \,\mathrm{d}s - \frac{1}{h}\int_0^h T(s)x \,\mathrm{d}s \\ &\to T(t)x - T(0)x = T(t)x - x.\end{aligned}$$

(3) Since $x \in D(A)$, by definition

$$\frac{T(t+h)x - T(t)x}{h} = T(t)\frac{T(h)x - x}{h} \to T(t)Ax. \qquad (*)$$

On the other hand, by commutativity

$$T(t)\frac{T(h)x - x}{h} = \frac{T(h)T(t)x - T(t)x}{h}, \qquad (**)$$

so the existence of limit implies $T(t)x \in D(A)$. But then

$$\lim_{h \to 0} \frac{T(h)T(t)x - T(t)x}{h} = AT(t)x. \tag{***}$$

Combining (*) and (***) we obtain the original claim.

(4) Integrate (3). □

Theorem 1.10

Let T be a C_0 semigroup and let A be its infinitesimal generator. Then $D(A) \subset X$ is dense and A('s graph) is closed (i.e., if $\{x_n\} \subset D(A)$ are such that $x_n \to x$ and $Ax_n \to y$ for some x, y, then $Ax = y$).

Proof. Given $x \in X$, we consider the approximation

$$x_t := \frac{1}{t} \int_0^t T(s)x \, \mathrm{d}s.$$

From Theorem 1.9(2) we know $x_t \in D(A)$, and by 1.9(1) we know $x_t \to x$ as $t \to 0$. This proves the density part.

Now to prove that the graph is closed, suppose $\{x_n\} \subset D(A)$ with $x_n \to x$, $Ax_n \to y$ for some x, y. Using 1.9(4),

$$T(t)x_n - x_n = T(t)x_n - T(0)x_n = \int_0^t T(s)Ax_n \, \mathrm{d}s.$$

Since $T(t)$ is bounded, as $n \to \infty$ we have $T(t)x_n - x_n \to T(t)x - x$. Also by assumption $Ax_n \to y$ so $\|Ax_n\|$ is uniformly bounded, say by K. Therefore, further using Theorem 1.8(1) to bound $\|T(s)\|$ gives

$$\int_0^t \|T(s)Ax_n\| \, \mathrm{d}s \leqslant \int_0^t \|T(s)\|\|Ax_n\| \, \mathrm{d}s \leqslant \int_0^t Me^{\omega s} K \, \mathrm{d}s < \infty.$$

By Lebesgue's DCT, $\int_0^t T(s)Ax_n \, \mathrm{d}s \to \int_0^t T(s)y \, \mathrm{d}s$. Putting everything together and dividing by t,

$$\frac{T(t)x - x}{t} = \frac{1}{t} \int_0^t T(s)y \, \mathrm{d}s.$$

Let $t \searrow 0$. By Theorem 1.9(1) again the RHS exists and $\to y$, so the limit of the LHS exists. By definition this means $x \in D(A)$ with the limit being Ax. So $Ax = y$ and A is closed. □

Theorem 2.1

Let T be a C_0 semigroup and let A be its infinitesimal generator. Then $\bigcap_{n \geqslant 1} D(A^n)$ is also dense in X.

Lemma 2.2

Let T be a C_0 semigroup and let A be its infinitesimal generator such that $\|T(t)\| \leqslant M$ for all $t \geqslant 0$ (i.e., the corresponding $\omega = 0$). Then if $x \in D(A)^2$ we have $\|Ax\|^2 \leqslant 4M^2 \|A^2 x\| \|x\|$.

Proof. Using Theorem 1.9(4), integration by parts, and 1.9(3),

$$\begin{aligned} T(t)x - x = T(t)x - T(0)x = \int_0^t T(s)Ax \, \mathrm{d}s &= -\int_0^t s(T(s)Ax)' \, \mathrm{d}s + tT(t)Ax \\ &= -\int_0^t sT(s)A^2x \, \mathrm{d}s + tT(t)Ax. \end{aligned} \tag{*}$$

Similarly,

$$T(t)Ax - Ax = \int_0^t T(s)A^2x \, \mathrm{d}s,$$

so multiplying by t and rearranging gives

$$0 = tAx - tT(t)Ax + t\int_0^t T(s)A^2x \, \mathrm{d}s. \tag{**}$$

Adding (*) and (**) and dividing by t gives

$$\frac{T(t)x - x}{t} = Ax + \int_0^t \frac{t-s}{t} T(s)A^2x \, \mathrm{d}s. \tag{***}$$

Hence

$$\begin{aligned} \|Ax\| &= \left\| \frac{T(t)x - x}{t} - \int_0^t \frac{t-s}{t} T(s)A^2x \, \mathrm{d}s \right\| \\ &\leqslant \left\| \frac{T(t)x - x}{t} \right\| + \left\| \int_0^t \frac{t-s}{t} T(s)A^2x \, \mathrm{d}s \right\| \\ &\leqslant \frac{\|T(t)x\| + \|x\|}{t} + \int_0^t \frac{t-s}{t} \|T(s)A^2x\| \, \mathrm{d}s \\ &\leqslant \frac{2M}{t} + \int_0^t \frac{t-s}{t} M\|A^2x\| \, \mathrm{d}s = \frac{2M}{t} + \frac{Mt\|A^2x\|}{2} \qquad \text{for all } t > 0. \end{aligned}$$

If $A^2x = 0$ then $\|Ax\| \leqslant 0$ and everything trivially holds. If $A^2x \neq 0$, we set

$$t := 2\left(\frac{\|x\|}{\|A^2x\|}\right)^{1/2}$$

so that

$$\|Ax\|^2 \leqslant \left(M\|x\|^{1/2}\|A^2x\|^{1/2} + M\|x\|^{1/2}\|A^2x\|^{1/2}\right)^2 = 4M^2\|x\|\|A^2x\|.$$

□

Two Examples

Example 2.3. Let $X := \{f \in C(\mathbb{R}) : f \text{ bounded and uniformly continuous}\}$ equipped with $\|\cdot\|_{\sup}$. Define

$$[T(t)f](x) := f(x+t).$$

Then $\|T\| = 1$ and such T forms a C_0 semigroup.

However, T is *not* a uniformly continuous semigroup. Consider f_n to be the function that equals 1 on $(-\infty, 0]$, 0 on $(1/n, \infty)$, and linear in-between. If T is uniformly continuous then

$$\|T(t)f_n - f_n\| = \|T(t)f_n - T(0)f_n\| \leqslant c(t) \qquad \text{for some } c(t) \to 0 \text{ as } t \to 0.$$

However, for each $t > 0$, there exists n sufficiently large so that $\|T(t)f_n - f_n\| = 1$. (To be precise, we need $1/n \leqslant t$ so that the tilted parts do not overlap even after being shifted by t.)

On the other hand,

$$Af = \lim_{h\to 0} \frac{f(\cdot + h) - f(\cdot)}{h} = f'$$

so $D(A) = \{f \in X : f^{-1} \text{ exists and } f^{-1} \in X\}$. In particular, $A^2 f = f''$ whenever it exists, so by Lemma 2.2

$$\|f'\|_{\sup}^2 \leqslant 4\|f''\|_{\sup}\|f\|_{\sup} \qquad \text{for all } f \in C^2(\mathbb{R}),$$

and by Theorem 2.2,

$$\bigcap_{n\geqslant 1}\{f : f^{(n)} \text{ exists and } \in X\} = \{f : f^{(n)} \text{ exists and} \in X \text{ for all n}\}$$

is still dense in X.

Example 2.4. We now consider $X := L^p(\mathbb{R})$ and $[T(t)f](x) := f(x+t)$ so that $\|T(t)\| = 1$. Then T is a C_0 semigroup by continuity of L^p norm with respect to translation, but it is again *not* a uniformly continuous semigroup.
Given t, we can take $f \in L^p$ with $\|f\|_p = 1$ such that all its "mass" is between 0 and t. Then

$$\|T(t)f - f\|_p = 2^{1/p}\|f\|_p \nrightarrow 0 \text{ as } t \to 0.$$

Here again $Af = f'$. We then have $\|f'\|_p^2 \leqslant 4\|f''\|_p\|f\|_p$ and that

$$\bigcap_{k\geqslant 0} W^{k,p} \qquad \text{is dense in } L^p.$$