Definition 2.5

Let T be a C_0 semigroup. We say

(1) T is **uniformly bounded** if $\|T(t)\| \leqslant M$ for some M and all t, and we say

(2) T is a **contraction** if $\|T(t)\| \leqslant 1$ for all t.

Theorem 2.6: Hille-Yosida

A is the infinitesimal generator of $T(t)$, a C_0 semigroup of contractions, if and only if:

(1) $D(A)$ is dense in X and A is closed, and

(2) The resolvent $\rho(A) := \{\lambda \in \mathbb{C} : |\lambda I - A|^{-1} \in B(X)\} \supset \mathbb{R}^+$ and $\|(\lambda I - A)^{-1}\| \leqslant 1/\lambda$ for all $\lambda > 0$.

Proof of $\Rightarrow$. We have proven (1) in Theorem 1.10. For the second part, for $\lambda > 0$, we define the resolvent by

$$R_\lambda x := \int_0^\infty e^{-\lambda t} T(t)x \, \mathrm{d}t.$$

(*This is well-defined because contraction gives* $\|T(t)x\| \leqslant \|x\|$.) First note that R_λ is bounded:

$$\|R_\lambda x\| \leqslant \int_0^\infty e^{-\lambda t}\|T(t)x\| \, \mathrm{d}t \leqslant \frac{\|x\|}{\lambda}$$

We now want to show that $R_\lambda \equiv (\lambda I - A)^{-1}$. Since

$$\begin{aligned}
\frac{T(h) - I}{h} R_\lambda x &= \frac{1}{h}\int_0^\infty e^{-\lambda t}(T(t+h) - T(t))x \, \mathrm{d}t \\
&= \frac{e^{\lambda h}}{h}\int_0^\infty e^{-\lambda(t+h)} T(t+h)x \, \mathrm{d}t - \frac{1}{h}\int_0^\infty e^{-\lambda t} T(t)x \, \mathrm{d}t \\
&= \frac{e^{\lambda h}}{h}\int_h^\infty e^{-\lambda t} T(t)x \, \mathrm{d}t - \frac{1}{h}\int_0^\infty e^{-\lambda t} T(t)x \, \mathrm{d}t \\
&= \frac{e^{\lambda h} - 1}{h}\int_0^\infty e^{-\lambda t} T(t)x \, \mathrm{d}t - \frac{e^{\lambda h}}{h}\int_0^h e^{-\lambda t} T(t)x \, \mathrm{d}t.
\end{aligned}$$

As $t \to 0$, we have $e^{\lambda h} \to 1$ and

$$\frac{1}{h}\int_0^h e^{-\lambda t} T(t)x \, \mathrm{d}t \to e^{-\lambda \cdot 0} \cdot T(0)x = x.$$

Also, $\frac{e^{\lambda h} - 1}{h} \to \lambda$. Therefore

$$\lim_{h \to 0} \frac{T(h) - I}{h} R_\lambda x = \lambda R_\lambda x - x.$$

That is, $R_\lambda x \in D(A)$, and

$$A R_\lambda x = \lambda R_\lambda x - x \iff (\lambda I - A) R_\lambda x = x \qquad \text{for all } x \in X. \tag{1}$$

It remains to show that $R_\lambda(\lambda I - A)x = x$ for all $x \in D(A)$. Note that for all $x \in D(A)$,

$$\begin{aligned}
R_\lambda A x &= \int_0^\infty e^{-\lambda t} T(t)Ax \, \mathrm{d}t = \int_0^\infty e^{-\lambda t} A T(t)x \, \mathrm{d}t && \text{(Theorem 1.9(3))} \\
&= \int_0^\infty A(e^{-\lambda t} T(t)x) \, \mathrm{d}t \\
&= A\left(\int_0^\infty e^{-\lambda t} T(t)x \, \mathrm{d}t\right) = A R_\lambda x && \text{(by closedness of } A\text{[?]).} \qquad (2)
\end{aligned}$$

This finishes the proof of $\Rightarrow$ ☐

Before proving $\Leftarrow$, we need some technical lemmas:

Lemma 2.7

Let A satisfy (1) and (2) in the proof above. Then $R_\lambda := (\lambda I - A)^{-1}$ satisfies

$$\lambda R_\lambda x \to x \qquad \text{as } \lambda \to \infty.$$

Proof. By assumption, for $x \in D(A)$, we have $\lambda R_\lambda x - x = AR_\lambda x = R_\lambda Ax$. We also know $\|(\lambda I - A)^{-1}\| \leqslant 1/\lambda$. Hence

$$\|\lambda R_\lambda x - x\| = \|R_\lambda Ax\| \leqslant \frac{\|Ax\|}{\lambda} \to 0$$

as $\lambda \to \infty$. If $x \notin D(A)$, we can pick a sequence $\{x_n\} \subset D(A)$ by density such that $x_n \to x$. Then,

$$\|rR_\lambda x - x\| \leqslant \|\lambda R_\lambda x_n - x_n\| + \|(\lambda R_\lambda - I)(x_n - x)\|.$$

Not that $\|R_\lambda\| \leqslant 1/\lambda$ so $\|\lambda R_\lambda - I\| \leqslant 2$. For $\|x_n - x\|$ sufficiently small, we obtain $\|\lambda R_\lambda x - x\| \to 0$. □

Definition 2.8: Yosida Approximation

$A_\lambda := \lambda AR_\lambda = \lambda^2 R_\lambda - \lambda I$ is called the **Yosida approximation**.

Lemma 2.9

$A_\lambda x \to Ax$ as $\lambda \to \infty$ for all $x \in D(A)$.

Proof. For $x \in D(A)$, using definition and Lemma 2.7, we have

$$A_\lambda x = \lambda AR_\lambda x = \lambda R_\lambda Ax \to Ax.$$

□

Lemma 2.10

A_λ is the infinitesimal generator of a uniformly continuous semigroup of contractions e^{tA_λ} for all $\lambda > 0$, and

$$\|e^{tA_\lambda}x - e^{tA_\mu}x\| \leqslant t\|A_\lambda x - A_\mu x\| \qquad \text{for all } t, \lambda, \mu > 0 \text{ and } x \in X.$$

Proof. Since $A_\lambda \in B_(X)$, it is a generator by Theorem 1.3. Since

$$\begin{aligned}\|e^{tA_\lambda}\| &= \|e^{-\lambda tI + \lambda^2 tR_\lambda}\| = \|e^{-\lambda t}e^{\lambda^2 tR_\lambda}\| \\ &= e^{-\lambda t}\|e^{t\lambda^2 R_\lambda}\| \leqslant e^{-\lambda t}e^{t\lambda^2\|R_\lambda\|}.\end{aligned}$$

Recall that $\lambda\|R_\lambda\| \leqslant 1$, and after cancellation everything is bounded by 1.

Finally, by FTC

$$\|e^{tA_\lambda}x - e^{tA_\mu}x\| = \left\| \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}(e^{tsA_\lambda}e^{t(1-s)A_\mu})\,\mathrm{d}s \right\| \leqslant t\|A_\lambda x - A_\mu x\|.$$

□

Proof of $\Leftarrow$ of Hille-Yosida. For all $x \in D(A)$, we have

$$\|e^{tA_\lambda}x - e^{tA_\mu}x\| \leqslant t\|A_\mu x - A_\lambda x\| \leqslant t\left(\|A_\mu x - Ax\| + \|A_\lambda x - Ax\|\right) \to 0$$

as $\mu, \lambda \to \infty$ by Lemma 2.9. That is, $\{e^{tA_\lambda}\}_{\lambda>0}$ is Cauchy, for all $t > 0$ and $x \in D(A)$.
Now we define

$$T(t) := \lim_{\lambda\to\infty} e^{tA_\lambda}x$$

(note that this limit is uniform on any compact $[0, T]$). It remains to show that $T(t)$ is a C_0 semigroup, i.e., $T(t)T(s) = T(t + s)$, with A being its infinitesimal generator.
Since

$$T(t)x - x = \lim_{n\to\infty}\left(e^{tA_\lambda}x - x\right) = \int_0^t e^{sA_\lambda}A_\lambda x \,\mathrm{d}s = \int_0^t T(s)Ax \,\mathrm{d}s,$$

we have the difference quotient

$$\frac{T(t)x - x}{t} = \frac{1}{t}\int_0^t T(s)Ax \,\mathrm{d}s \to T(0)Ax = Ax$$

for all $x \in D(A)$, by Theorem 1.9(1).
If B is the infinitesimal generator o $T(t)$, then above gives $D(A) \subset D(B)$ and $A = B$ on $D(A)$.
Note that the image of $I - B$ is all of X, since $(I - B)^{-1} = (I - 1 \cdot B)^{-1} \in B(X)$, which is given by the second assumption on resolvent. Therefore, $D(B)$, $D(I - B)$, and the image of $(I - B)^{-1}$ are all the same.
On the other hand, the image of $I - A = X$, by the assumption on resolvent again. Since the image is clearly defined only on $D(A)$, we have $(I - A)D(A) = X$, and since $A = B$ on $D(A)$, we have

$$(I - A)D(A) = (I - B)D(A) = X \implies D(A) = (I - B)^{-1}X = D(B).$$

Therefore A and B have the same domain and equals each other on the domain, and we are done. □

Corollary 3.1

Let A be the infinitesimal generator of T, a C_0 semigroup of contractions. Then

$$\rho(A) \supset \{\lambda : \mathfrak{Re}\lambda > 0\} \qquad \text{and} \qquad \|R_\lambda\| \leqslant 1/\lambda.$$

Example 3.2. Let X be the collection of bounded and uniformly continuous functions on $(0, \infty)$ and define $T(t)f := f(\cdot + t)$. Then T is a C_0 semigroup of contractions.
Let $D(A) := \{f : f, f' \in X\}$ and $Af := f'$. Then $\rho(A) \supset \{\lambda : \mathfrak{Re}\lambda > 0\}$ by the previous corollary.
If $\mathfrak{Re}\lambda \leqslant 0$, then $\varphi_\lambda(s) := e^{\lambda s} \in X$ satisfies $(\lambda I - A)\varphi_\lambda = 0$.

Corollary 3.3

A is the infinitesimal generator of $T(t)$, a C_0 semigroup, such that $\|T(t)\| \leqslant e^{\omega t}$ ($\omega \geqslant 0$), if and only if

(1) A is dense and $\overline{D(A)} = X$, and

(2) $\rho(A) \supset \{\lambda : \mathfrak{Re}\lambda > \omega\}$ and $\|R_\lambda\| \leqslant 1/(\lambda - \omega)$.

Proof. We define $S(t) := e^{-\omega t}T(t)$ so offset the effects of ω and then apply Hille-Yosida. $\square$

Definition 3.4: Dissipative Operators

Let A be a linear operator from $D(A)$ to X. We say A is **dissipative** if, for all $x \in D(A)$, there exists $f \in F(x) := \{f \in X^* : f(x) = \|x\|^2 = \|f\|^2_{X^*}\}$ (the *duality set*), with

$$\mathfrak{Re} f(Ax) \leqslant 0.$$

(Note by Hahn-Banach, given x, there exists $f \in X^*$ with $\|f\| = 1$ and $f(x) = \|x\|$, so $\|x\|f \in F(x)$ and in particular the duality set is always nonempty.)

Theorem 3.5

A is dissipative if and only if

$$\|(\lambda I - A)x\| \geqslant \lambda\|x\| \qquad \text{for all } x \in A.$$

Theorem 3.6: Lumer-Philips

Let A be a linear operator from $D(A)$ to X, where $D(A) \subset X$ is dense. Then

(1) If A is dissipative and $\lambda_0 I - A$ is surjective for some $\lambda_0 > 0$, then A generates a C_0 semigroup of contractions, and

(2) Conversely, if A is the infinitesimal generator of T, a C_0 semigroup of contractions, then A is dissipative and $\lambda I - A$ is subjective for *all* $\lambda > 0$. In addition, $\mathfrak{Re} f(Ax) \leqslant 0$ for *all* $x \in D(A)$ and $f \in F(x)$.

Example 3.7: [?]. Define $A(x, D) := \sum_{|\alpha|=2m}^{a_\alpha(x)} D^\alpha$. ($A = \Delta$ when $m = 1$ for example.) Then A is **strongly elliptic** if there exists $c > 0$ such that

$$\mathfrak{Re}(-1)^m A(x, \xi) \geqslant c|\xi|^{2m}.$$

We also have the **Gårding inequality**:

$$\mathfrak{Re}\,\langle Au, u\rangle_{L^2} \geqslant c_0\|a\|^2_{H^m} - \lambda_0\|u\|^2_{L^2} \qquad \text{for some } c_0, \lambda_0 \text{ constant.}$$