Beginning of May 24, 2022

Convex Sets

Recall that a set C is **convex** if, for all $x_1, x_2 \in C$ and $\theta \in [0,1]$,

$$\theta x_1 + (1-\theta)x_2 \in C.$$

We also define the **convex hull** of C, written conv C, to be the set of all convex combinations of points in C:

$$\text{conv } C := \{\theta_1 x_1 + \ldots + \theta_k x_k : x_i \in C, \theta_i \geq 0, \sum \theta_i = 1\}.$$

(Similarly, a set C is **affine** if, for all $x_1, x_2 \in C$ and $\theta \in \mathbb{R}$, $\theta x_1 + (1-\theta)x_2 \in C$.)
More generally, we can generalize the notion of convex sets into infinite sums or integrals / probability distributions. Given a set C, we define the **conic hull** of C to be the set of all **conic combinations** of C:

$$\{\theta_1 x_1 + \ldots + \theta_k x_k : x_i \in C, \theta_i \geq 0\}.$$

This also happens to be the smallest cone containing C.

0.1 Examples of Convex Sets

Cones

A set C is a **cone** if for every $x \in C$ and $\lambda \geq 0$, we have $\lambda x \in C$. A **convex cone** is a convex cone... One simple example is $\{x \in \mathbb{R}^3 : x_1^2 + x_2^2 = x_3^2\}$. It should be clear that any such homgogeneous equation describes a cone, but not vice versa.

Theorem 0.1.1

C is a convex cone if and only if for any $x_1, x_2 \in C$, $\lambda_1 x_1 + \lambda_2 x_2 \in C$ for $\lambda_1, \lambda_2 \geq 0$.

Proof. Suppose C is a convex cone. Then by the cone assumption $\lambda_1 x_1, \lambda_2 x_2 \in C$. Therefore $\lambda_1 x_1/2 + \lambda_2 x_2/2 \in C$ by convexity, and $2(\lambda_1 x_1/2 + \lambda_2 x_2/2) \in C$ by the cone assumption again.
Conversely, for $x_1, x_2 \in C$ and $\lambda \in [0,1]$, letting $\lambda_1 := \lambda$ and $\lambda_2 := 1-\lambda$ shows the claim. □

Hyperplanes

A **hyperplane** is a set of form $\{x : a^T x = b\}$ where $0 \neq a \in \mathbb{R}^n$ and $b \in \mathbb{R}$. Alternatively, this can be represented as $\{x : a^T(x - x_0) = 0\}$ where x_0 is any solution satisfying $a^T x_0 = b$. Hence the hyperplane is some plane with normal vector a.

Euclidean Balls and Ellipsoids

An **Euclidean ball** in $\mathbb{R}^n$ is a ball of form

$$B(x_0, r) := \{x : \|x - x_c\|_2 \leq r\} = \{x : (x-x_0)^T(x-x_0) \leq r^2\}.$$

This notion can be extended to other norms. An **ellipsoid** centered at c is of form

$$\mathcal{E} := \{x : (x-c)^T P^{-1}(x-c) \leqslant 1\}$$

where P is a symmetric PD matrix. The lengths of the semi-axes of $\mathcal{E}$ are $\sqrt{\lambda_i}$, the roots of P's eigehvalues. This can be expressed explicitly as the set of points satisfying of form

$$\frac{(x_1-c_1)^2}{\lambda_1} + \cdots + \frac{(x_n-c_n)^2}{\lambda_n} \leqslant 1.$$

To show $\mathcal{E}$ is convex, we WLOG assume $P = D^{-1}$ is diagonal since this can always be achieved by a change of basis. Also we may assume $c = 0$. Then the ellipsoid is simply $x^T Dx = \|D^{1/2}x\| \leqslant 1$. By linearity, triangle inequality, and homogeineity of norm, the original claim follows from the fact that

$$\|D^{1/2}(\lambda x_1 + (1-\lambda)x_2)\| \leqslant \lambda\|D^{1/2}x_1\| + (1-\lambda)\|D^{1/2}x_2\| \leqslant \lambda + (1-\lambda) = 1.$$

Yet another equivalent definition of an ellipsoid is given by

$$\mathcal{E} := \{c + Au : \|u\|_2 \leqslant 1\}$$

where A is square and nonsingular. In this case, $A = P^{1/2}$.

Norm Balls and Norm Cones

Suppose $\|\cdot\|$ is a norm on $\mathbb{R}^n$. The **norm cone** associated with the norm $\|\cdot\|$ is

$$C := \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : \|x\| \leqslant t\} \subset \mathbb{R}^{n+1}$$

is a convex cone. Using the previous characterization of convex cones, for (x_1, t_1) and (x_2, t_2) in the cone and $\theta_1, \theta_2 \geqslant 0$, we have

$$\|\theta_1 x_1 + \theta_2 x_2\| \leqslant \theta_1\|x_1\| + \theta_2\|x_2\| \leqslant \theta_1 t_1 + \theta_2 t_2.$$

Polyhedra

A **polyhedron** is defined to be as the set of points satisfying linear equalities/inequalities:

$$\mathcal{P} := \{x : a_j^T x \leqslant b_j \text{ for } 1 \leqslant j \leqslant m \text{ and } c_j^T x = d_j \text{ for } 1 \leqslant j \leqslant p\}.$$

Writing this in forms of matrix, we have $\mathcal{P} = \{x : Ax \preceq b, Cx = d\}$. Convexity follows immediately from linearity of matrix operations.

Simplex

If $v_0, \ldots, v_k \in \mathbb{R}^n$ are affinely independent ($v_1 - v_0, \ldots, v_k - v_0$ are linearly independent), then the correspoinding k-dimensional **simplex** determined by these points is the convex hull $C := \text{conv}\,\{v_0, \ldots, v_k\}$.

There are two special types of simplexes. The **unit simplex** in $\mathbb{R}^d$ is generated by $\{0, e_1, \ldots, e_d\}$:

$$\{x \in \mathbb{R}^d : x \succeq 0 \text{ and } \|x\|_1 = 1\}.$$

The (($d-1$)-dimensional) **probability simplex** is generated by $\{e_1, \ldots, e_d\}$ instead.

Theorem 0.1.2

If $\mathcal{P}$ is a (finite-dimensional) simplex then it is a polyhedron. Conversely, if $\mathcal{P}$ is a polyhedron can be represented as

$$\{\theta_1 x_1 + \ldots + \theta_m x_m + \ldots + \theta_k x_k : \theta_i \geqslant 0, \sum_{i=1}^{m} \theta_i = 1\},$$

namely, the set theoretic sum of a simplex and a cone, which accounts for "unbounded polyhedra."

Main idea: we can transform any simplex into a unit simplex of proper dimension and use the fact that unit simplexes are polyhedra.

The PSD Cone

For notational convenience, we use $\mathbb{S}^n$ to denote the set of $n \times n$ real symmetric matrices, which is of dimension $n(n+1)/2$. Let $\mathbb{S}^n_+$ denote the set of PSD symmetric matrices and $\mathbb{S}^n_{++}$ the set of PD symmetric matrices. The set $\mathbb{S}^n_+$ is a convex cone: if A, B are PSD symmetric matrices and $\theta_1, \theta_2 \geqslant 0$, then

$$x^T(\theta_1 A + \theta_2 B)x = \theta_1 x^T Ax + \theta_2 x^T Bx \geqslant 0.$$

0.2 Operations that Preserve Convexity

Intersection

Convexity is preserved under *arbitrary* intersection. For example, a polyhedron is the intersection of halfspaces and hyperplanes and is therefore convex.

Images under Affine Transformations

If $f : \mathbb{R}^n \to \mathbb{R}^m$ is affine then it is of form $f(x) = Ax + b$. If $C \subset \mathbb{R}^n$ is convex, then the image $f(S)$ is convex in $\mathbb{R}^m$ as well. For example, an ellipse can be obtained from a ball under an affine map and is therefore convex. The *inverse image* is affine as well, even when the map is not injective and the inverse undefined.

Projections

Suppose $C \subset \mathbb{R}^d$ is convex and V is a subspace of $\mathbb{R}^n$. Then the projection of C onto V is convex, without any assumptions on whether the projection is orthogonal. Note that projections are a special case of linear transformations.

Products and Sums

If $C_1, \ldots, C_k$ are convex sets in $\mathbb{R}^d$, then the Cartesian product $C_1 \times \ldots \times C_k$ and also the sum $\sum C_i$ are both confex.