MATH 625 Homework 1

Qilin Ye

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Problem 1: (Boyd 2.2)

Show that a set is convex if and only if its intersection with any line is convex. Show that a set is affine if and only if its intersection with any line is affine.

Proof. Let C be convex and let $\{x + kv : k \in \mathbb{R}\}$ be a line. If the intersection is empty then the claim trivially holds. If the intersection is nonempty, let $x + k_1 v, x + k_2 v \in C$ and let $\lambda \in (0, 1)$. By convexity,

$$\lambda(x + k_1 v) + (1 - \lambda)(x + k_2 v) = x + (\lambda k_1 + (1 - \lambda)k_2)v \in C \tag{*}$$

and clearly such point is still in the intersection. The converse follows from the same reasoning backwards. The affine case is highly identical, except we allow λ to be any real number. □

Problem 2: (Boyd 2.3)

A set C is **midpoint convex** if whenever two points $a, b \in C$, the midpoint $(a + b)/2 \in C$. Obviously a convex set is midpoint convex. It can be proved that under mild conditions midpoint convexity implies convexity. As a simple case, prove that if C is closed and midpoint convex, then C is convex.

Proof. Suppose C is not convex, so there exist $x, y \in C$ and $\lambda \in (0, 1)$ such that $\lambda x + (1 - \lambda)y \notin C$. We can express λ as $\lambda = \sum_{n \geqslant 1} c_n 2^{-n}$ where c_n is the n^{th} digit of the binary expansion of λ after the decimal point. By applying midpoint convexity recursively and using the assumption that C is closed, we see

$$\lim_{n \to \infty} \left[\sum_{k=1}^{n} c_n 2^{-n} \cdot x + (1 - \sum_{k=1}^{n} c_n 2^{-n}) \cdot y \right] = \lambda x + (1 - \lambda)y \in C.$$ □

Problem 3

Let $C \subset \mathbb{R}^d$ and let λ_1, λ_2 be positive scalars. Show that if C is convex then $(\lambda_1 + \lambda_2)C = \lambda_1 C + \lambda_2 C$. Show by example that this may not be true if C is not convex.

Proof. If $x \in (\lambda_1 + \lambda_2)C$ then there exists $x_0 \in C$ such that $x = (\lambda_1 + \lambda_2)x_0$. Then $\lambda_1 x_0 \in \lambda_1 C$ and $\lambda_2 x_0 \in \lambda_2 C$. Conversely, suppose $x = \lambda_1 c_1 + \lambda_2 c_2$ where $c_1, c_2 \in C$. Then

$$x = \lambda_1 c_1 + \lambda_2 c_2 = (\lambda_1 + \lambda_2) \left[\frac{\lambda_1}{\lambda_1 + \lambda_2} \cdot c_1 + \frac{\lambda_2}{\lambda_1 + \lambda_2} \cdot c_2 \right]$$

and the term in square brackets is in C by convexity. This completes the proof.

If C is not convex, however, consider $x \neq 0$ and $C := \{0, x\}$. Let $\lambda_1 = \lambda_2 = 1$ and we have $(\lambda_1 + \lambda_2)C = \{0, 2x\}$ and $\lambda_1 C + \lambda_2 C = \{0, x, 2x\}$. □

Problem 4

Prove a convex set is connected, path connected, and simply connected.

Proof. If C is convex and $x, y \in C$, the line segment joining x and y is always contained in C so C is (path) connected.

To show C is simply connected, let $x, y \in C$ and suppose $f, g : [0,1] \to C$ are two curves with endpoints $f(0) = g(0) = x$ and $f(1) = g(1) = y$. The function

$$H(t, \lambda) := \lambda f(t) + (1 - \lambda) g(t)$$

is continuous, with $H(\cdot, 0) \equiv g$, $H(\cdot, 1) \equiv f$, $H(0, \cdot) \equiv x$, and $H(1, \cdot) \equiv y$. Therefore any two paths connecting the same two endpoints are homotopic and therefore C is simply connected. □

Problem 5

Prove that if C is compact, $x \in \operatorname{relint}(C)$, and $y \in \overline{C}$, then the line segment $[x, y)$ is contained in C.

Proof. WLOG assume the affine hull of C is the entire space so that $x \in \operatorname{relint}(C)$ directly translates to "there exists $r > 0$ such that $B(x, r) \subset C$." For $\epsilon > 0$, there exists $y_\epsilon \in C$ with $\|y - y_\epsilon\| < \epsilon$. For any $\lambda \in (0, 1)$, define $\delta := \lambda r - (1 - \lambda)\epsilon$ and WLOG assume ϵ is sufficiently small so that $\delta > 0$. Then, if $z \in B(\lambda x + (1 - \lambda) y, \delta)$, we have

$$\begin{aligned} \|z - (\lambda x + (1 - \lambda) y)\| &\leqslant \|z - (\lambda x + (1 - \lambda) y_\epsilon)\| + (1 - \lambda)\|y - y_\epsilon\| \\ &< \|z - (\lambda x + (1 - \lambda) y_\epsilon)\| + (1 - \lambda)\epsilon < \lambda r. \end{aligned}$$

Since $B(x, r) \subset C$ and $y_\epsilon \in C$, convexity implies $z \in C$, so $B(\lambda x + (1 - \lambda) y, \delta) \subset C$. Therefore, we've shown that every point on the line segment is in fact contained in the interior of C. □

Problem 6

Suppose $A \subset \mathbb{R}^d$ and H is a hyperplane in $\mathbb{R}^d$. Suppose A is contained in one of the half-spaces defined by H. Prove that

$$\operatorname{conv}(H \cap A) = H \cap \operatorname{conv}(A).$$

Proof. Let the hyperplane be characterized by $\{x : v^T x = b\}$ and WLOG assume all $a \in A$ satisfy $v^T a \geqslant b$. If $H \cap A = \varnothing$ there is nothing to show, since if $a_1, \ldots, a_k \in A$ and $\sum_{i=1}^{k} \theta_i = 1$ with $\theta_i \geqslant 0$, then

$$v^T \left(\sum_{i=1}^{k} \theta_i a_i\right) = \sum_{i=1}^{k} \theta_i v^T a_i > \sum_{i=1}^{k} \theta_i b = b, \tag{*}$$

so $H \cap \operatorname{conv}(A) = \varnothing$ as well.

One direction is clear:

$$\begin{cases} \mathrm{conv}(H \cap A) \subset \mathrm{conv}(H) = H \\ \mathrm{conv}(H \cap A) \subset \mathrm{conv}(A) \end{cases} \quad \text{so} \quad \mathrm{conv}(H \cap A) \subset H \cap \mathrm{conv}(A).$$

Conversely, suppose $x \in H \cap \mathrm{conv}(A)$. By definition there exist $a_1, \dots, a_k \in A$ and $\sum_{i=1}^{k} \theta_i = 1$ with $\theta_i \geqslant 0$ such that $a = \sum_{i=1}^{k} \theta_i a_i$. Furthermore $a \in H$. From (*) we see that each a_i must satisfy $v^T a_i = b$, namely each $a_i \in H$ and in particular $a_i \in H \cap A$. The same formula for a used earlier completes the proof. □

Problem 7

Let C be a closed, bounded, convex set in $\mathbb{R}^d$. Let H be a supporting hyperplane of C. Then the intersection $H \cap C$ is called a **face** of C. Show that every such C has at least one 0-dimensional face.

Proof. Since C is closed and bounded in $\mathbb{R}^d$ it is compact. The norm function on C therefore attains at least one maximum, say at $a \in C$. Consider the hyperplane passing through and orthogonal to a, namely $H := \{x \in \mathbb{R}^d : (x - a)^T a = 0\}$. Since C is convex, $H \cap C = \{a\}$, for otherwise Pythagorean theorem implies a does not maximize the norm function on C. This completes the proof since a singleton is zero-dimensional. □

Problem 8: (Boyd 2.22)

Complete the proof of the separating hyperplane theorem.

Proof. We've shown this in class so I will only briefly state the key ideas. Denote $X := C - D$. If $0 \notin X$, we are done by the theorem on the special case. If $0 \in X$ but X is affine then the hyperplane containing X already serves as a separator and we are done.

Otherwise, for any $\epsilon > 0$, consider $X_\epsilon := \{x \in X : B(x, \epsilon) \subset X\}$, the "$\epsilon$-interior" of X. The theorem on special case implies that there exists a separating hyperplane for $\{0\}$ and X_ϵ. WLOG assume the hyperplane is characterized by a normal vector v_ϵ (e.g., $v_\epsilon^T x > b_\epsilon$ for some b_ϵ we don't care about). Let $\epsilon_n := 1/n$ and we obtain a sequence of unit vectors. By compactness of the unit ball in $\mathbb{R}^d$ there exists a convergent subsequence and a limit point $v \in B_1$. Since the ϵ_n's corresponding to the subsequence still converge to 0, we have $\lim_{n\to\infty} X_{\epsilon_n} = X$, thus completing the proof. □

Problem 9: (Boyd 2.29)

Suppose $K \subset \mathbb{R}^2$ is a closed convex cone.

(1) Give a simple description of K in polar coordinates.

(2) Give a simple description of K^* and draw a plot illustrating the relation between K and K^*.

(3) When is K pointed?

(4) When is K proper? Draw a plot illustrating what $x \preceq_K y$ means when K is proper.

Solution. (1) $K = \{(r\cos\theta, r\sin\theta) : r \geqslant 0, \theta \in [a,b]\}$ for some $[a,b]$ depending on the shape of the cone. (Here we assume k is closed.)

(2) (*Drawing in LaTeX is tough so I hope a verbal description suffices.*) K^* is also a cone. In polar coordinates, it can be represented as $\{(r\cos\theta, r\sin\theta) : r \geqslant 0, \theta \in [a^*, b^*]\}$ where

$$[a^*, b^*] := [a - \pi/2, a + \pi/2] \cap [b - \pi/2, b + \pi/2]$$

assuming the intersection is nonempty. For example if K is a halfspace, K^* becomes a degenerate ray perpendicular to the hyperplane corresponding to K.

(3) K is pointed when $b - a < \pi$ as described in the first part.

(4) K needs to be closed as assumed in the first part. It is trivially convex. To have nonempty interior we require $b > a$. To be pointed, see previous part. $x \preceq_K y$ means $y \in x + K$, i.e., y is in a "translated cone".

Problem 10: (Boyd 2.32)

Find the dual cone of $\{Ax : x \preceq 0\}$ where $A \in \mathbb{R}^{m\times n}$.

Solution. Since ($x^T A^T y \geq 0$ for all $x \geq 0$) if and only if $y \geq 0$, the dual cone is simply $K^* := \{y : A^T y \geq 0\}$.