MATH 625 Homework 3

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Problem 1

Let f be a convex function defined on $\mathbb{R}^n$. Show that x^* is global minimum if and only if it is a global minimum along every line passing through x^*. That is, for every $v \neq 0 \in \mathbb{R}^n$, the function $g(t) := f(x^* + tv)$ has $t = 0$ as its global minimum.

Proof. Global minimizer implying directional minimized is trivial. The converse is also trivial: if x^* minimizes f along all lines passing through x^* then clearly no other point y can have $f(y) < f(x^*)$ for we simply consider the line determined by the two points. Maybe there's a issue with the phrasing of the word? □

Problem 2

Let f be quasiconvex and let X be a convex set with $\operatorname{dom} f \cap X \neq \varnothing$. Suppose that f is not constant along any segment contained in X. Prove that any local minimum of f over X is also global.

Proof. Let $x \in X$ be a local minimum and suppose $y \in X$ satisfies $f(y) < f(x)$. Consider the sublevel set $S := \{x' \in X : f(x') \leqslant f(x)\}$ which, by assumption, is convex. Therefore $S \cap X$ is convex, and clearly x, y are both contained in the intersection. This means the line segment joining x and y is also in $S \cap X$, so in particular the value of f along this line $\leqslant f(x)$. Since f is nonconstant along any line segment, this implies that given $\epsilon > 0$ there exists z on the line with $d(x, z) < \epsilon$ and $f(z) < f(x)$. Since ϵ is arbitrary, we see f cannot have a local minimum at x, contradiction. □

Problem 3

Suppose you have a convex optimization problem with objective function f_0 and feasible set X with $\operatorname{dom} f_i = \mathbb{R}^n$ for all i. Prove that the set of optimal points X^* is convex. Show that if $\operatorname{dom} f_0 \cap X$ is nonempty and bounded then X^* is nonempty and compact.

Proof. If x_1 and x_2 are optimal points and $\lambda \in (0, 1)$, then by convexity, $\lambda x_1 + (1-\lambda)x_2$ also satisfies all constraint functions while $f_0(\lambda x_1 + (1-\lambda)x_2) \leqslant \lambda_0 f(x_1) + (1-\lambda)f(x_2)$. by assumption x_1 and x_2 are optimal, so the in equality must in fact attain equality. This proves X^* is convex.

For the second part it suffices to show that X^* is closed and nonempty. Let x be a limit point of X^* and consequently let $\{x_n\}$ be a sequence in X^* converging to x. By continuity,

$$f_i(x) = \lim_{n\to\infty} f_i(x_n) \leqslant 0$$

for $1 \leqslant i \leqslant n$. Similarly, by passing to limits, we see that f_0 also attains minimum at x. □

Problem 4: Boyd 4.1

Consider the optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x_1, x_2) \\ \text{subject to} & 2x_1 + x_2 \geqslant 1 \\ & x_1 + 3x_2 \geqslant 1 \\ & x_1 \geqslant 0, x_2 \geqslant 0. \end{array}$$

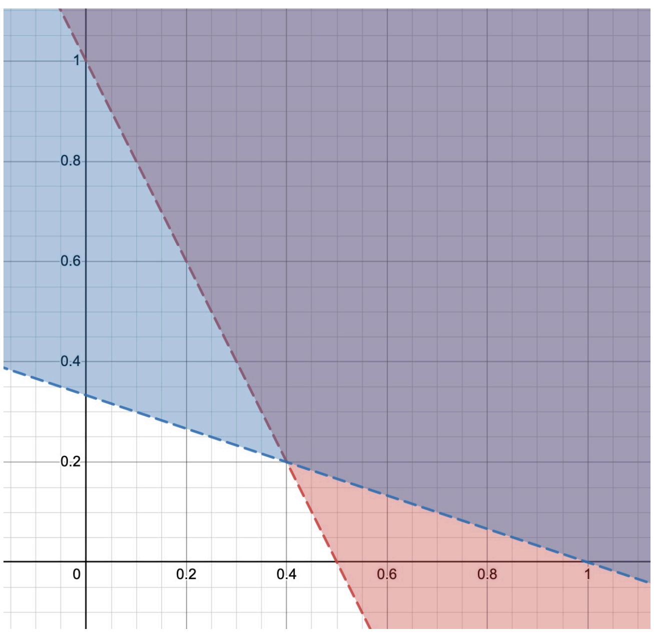
Make a sketch of the feasible set. For each of the following objective functions give the optimal set and the optimal value.

(a) $f_0(x_1, x_2) = x_1 + x_2$.

(c) $f_0(x_1, x_2) = x_1$.

(e) $f_0(x_1, x_2) = x_1^2 + 9x_2^2$.

Solution. The feasible set is just the first quadrant excluding the white, blue, red parts.



(a) The optimal set is the singleton containing the intersection of the two sloped lines, namely $\{(2/5, 1/5)\}$ with optimal value $3/5$.

(c) The optimal set is $\{0\} \times [1, \infty)$ with optimal value 0.

(e) The optimal point will take place on the boundary. f takes value 9 on $(0,1)$, 1 on $(1,0)$. We parametrize the line segment between $(0,1)$ and $(0.4,0.2)$ via $(t, 1-2t), t \in [0, 0.4]$ and find minimum value of f to be 0.52. We parametrize the other line segment between $(1,0)$ and $(0.4,0.2)$ by $(1-3t, t)$ for $t \in [0, 0.2]$ and find the minimum to be $1/2$ at $(1/2, 1/6)$. Therefore the optimal set is $\{(1/2, 1/6)\}$ with optimal value $1/2$.

4.2. (1) If there exists v with $Av \leq 0$ then certainly $\lambda Av \leq 0$ for all $\lambda > 0$. This implies $x + \lambda v \in \mathrm{dom}(f)$ for all $\lambda > 0$ and $x \in \mathrm{dom}(f)$, making the domain unbounded.

Conversely, suppose the domain is unbounded and let $\{x^{(n)}\}$ be a sequence with $\|x^{(n)}\|_2 \to \infty$. Since ∂B_1 is compact, the sequence $\{x^{(n)}/\|x^{(n)}\|_2\}$ has a convergent subsequence. The limit, which we call x, satisfies $\|x\|_2 = 1$ and $Ax \leq 0$. To see this, recall that for each n we have $Ax^{(n)} \leq b$, so

$$A(x^{(n)}/\|x^{(n)}\|_2) \leq b/\|x^{(n)}\|_2.$$

Passing the inequality to the aforementioned subsequence and letting the index tend to ∞, we have $Ax \leq 0$.

(2) If such v exists, then for $\lambda > 0$ and $x \in \mathrm{dom}(f)$,

$$f_0(x + \lambda v) = -\sum_{i=1}^{m} \log(b_i - a_i^T x - \lambda a_i^T v).$$

As λ increases, the terms $b_i - a_i^T x - \lambda a_i^T v$ increases. Letting $\lambda \to \infty$, the terms in the RHS $\to -\infty$, so f_0 is unbounded from below.

Conversely, suppose there exists a sequence $\{x^{(n)}\}$ with $(b - Ax^{(n)} \geq 0$, which we implicitly assume, and) $f_0(x^{(n)}) \to -\infty$. In particular, for some $j \in [1, m]$ we must have

$$\log(b_j - a_j^T x^{(n)}) \to \infty \implies \lim_{n\to\infty} \max_{1 \leqslant j \leqslant m} (b_j - a_j^T x^{(n)}) = \infty.$$

Using the hint, suppose there exists z with $z \succ 0$ and $A^T z = 0$. Then $(A^T z)^T x^n = 0$. Therefore

$$z^T(b - Ax^{(n)}) = z^T b - z^T A x^{(n)} = z^T b$$

whereas

$$z^T b = \sum_{i=1}^{m} z_i b_i \geqslant z_j b_j \to \infty.$$

(Here we abuse the notation, assuming that for a fixed m, $j := \mathrm{argmax}_j (b_j - a_j^T x^{(n)})$.) Contradiction,

(3) If the domain is bounded then the sublevel sets are closed and therefore compact. Pick any sublevel set, and f_0 attains a minimum on it, and this minimum must also be the global minimum of f_0.

If the domain is unbounded, it needs to be unbounded in some direction. Let v be any vector such that, for all $M > 0$, there exists a scalar multiple of v with norm $> M$. (That is, v is a "direction" along which $\mathrm{dom}(f)$ is unbounded.) It follows that $Av \leq 0$. By the previous part we must have $Av = 0$, so $f_0(\lambda v)$ is constant for all $\lambda > 0$. Excluding all such directions, we obtain a bounded set, so f_0 obtains a minimum on the remaining subset of $\mathrm{dom}(f)$. Therefore the minimum must be attained in either case.

(4) f is strictly convex so there can be at most one optimal point. □

Boyd 4.6. I believe (2) should be "nondecreasing" instead. Suppose x is optimal for (2) but $h(x) < 0$. We can slightly decrease the r^{th} component of x, resulting in $\tilde{x}$, with $h(\tilde{x}) \leqslant 0$. Meanwhile we also make sure the subjective function decreases and the constraints are followed. This means x is not optimal for (2). □

Boyd 4.8. (1) If infeasible, the answer is ∞. Otherwise decompose c as $A^T v + w$ where w is in the nullspace of A. Then

$$c^T x = v^T Ax + w^T x = v^T b + w^T x.$$

If $w = 0$ then the optimal value is simply $v^T b$. Otherwise, any solutions of form $x + \lambda w$ works, so the answer is $-\infty$.

(2) $a^T x \leqslant b$ always has a solution so the system is feasible. We decompose c according to a: $c = \lambda a + d$ where $\lambda \in \mathbb{R}$ and $d^T a = 0$. Then

$$c^T x = (\lambda a + d)^T x = \lambda a^T x + d^T x.$$

If $\lambda > 0$ then the problem is unbounded from below by considering $x = -ta$, $t \to \infty$, since $a^T x = -ta^T a \to -\infty$ and $c^T x = -ta^T a \to -\infty$.

If $\lambda = 0$ then d is perpendicular to a. (If $c = 0$ there is nothing to show.) By considering $x = ba - tc$, we have

$$c^T x = bd^T a - tc^T c = -tc^T c \to -\infty$$

and indeed

$$a^T (ba - tc) = ba^T a - ta^T c < b \qquad \text{eventually.}$$

If $\lambda < 0$ and $d = 0$ then $c = \lambda a$, so $c^T x = \lambda a^T x \geqslant \lambda b$.

Finally, if $\lambda < 0$ and $d \neq 0$ then using $x = ba - tc$ we have $c^T x \to \infty$ once more as $t \to \infty$. Therefore,

$$p* = \begin{cases} ca/b & \text{if } c/a \in \mathbb{R}_- \\ -\infty & \text{otherwise.} \end{cases}$$

(3) It suffices to minimize componentwise:

$$x_i^* := \begin{cases} l_i & c_i > 0 \\ u_i & c_i \leqslant 0. \end{cases}$$

(4) This is a weighted average problem with minimum attained when all x_i are 0 except the one corresponding to the smallest component of c. In this case $c^T x$ is exactly the value of that component.

If $1^T x \leqslant 1$ instead, we set all $x_i = 0$ if the smallest component of c is positive, or we keep the answer in the $1^T x = 1$ case if the smallest component is negative.

(5) Similar to the previous part, if α is an integer then the minimum value corresponds to the sum of the α smallest components of c. If α is not an integer, the minimum is the sum of the $\lfloor \alpha \rfloor$ smallest compoeenents of c, plus $(\alpha - \lfloor \alpha \rfloor)$ times the remaining smallest component.

If $1^T x \leqslant \alpha$ then we simply further require the chosen components to be nonpositive and replace the positive ones by 0.

(6) We instead consider

$$\text{minimize } \sum_{i=1}^{n} (c_i/d_i) y_i$$
$$\text{subject to } 1^T x = \alpha, 0 \le y \le d.$$

The result then follows from the previous part.

□

Boyd 4.9. Assuming A is invertible, minimizing $c^T x$ is equivalent to minimizing $c^T A^{-1} y = (A^{-T} c)^T y$ with constraint $y \le b$. If $A^{-T} c \le 0$ the optimal point is $y = b$ or $x = A^{-1} b$, so $p^* = c^T A^{-1} b$. Otherwise, letting $y \le 0$ and $t \to \infty$, we have $(A^{-T} c)(ty) \to -\infty$. □

Boyd 4.19. (a) We need to verify that the objective function is quasiconvex. The level sets are of form

$$\|Ax - b\|_1 / (c^T x + d) \geqslant t \implies \|Ax - b\|_1 - \alpha(c^T x + d) \leqslant 0.$$

It is clear that such set is convex by triangle inequality, so the problem is indeed a quasiconvex optimization.

(b) Suppose we have minimized $\|Ay - bt\|_1$ subject to $\|y\|_\infty \leqslant t$ and $c^T y + dt = 1$. It is clear that $t > 0$ for otherwise $y = 0$ and $c^T y + dt = 0$. If we define $x = y/t$ then $\|x\|_\infty \leqslant 1$, and

$$\frac{\|Ax - b\|_1}{c^T x + d} = \frac{\|Ax - b\|_1}{1/t} = \|Ay - bt\|_1.$$

Conversely suppose $\|x\|_\infty \leqslant 1$. Then if

$$y = x/(c^T x + d) \qquad \text{and} \qquad t = 1/(c^T x + d)$$

we have

$$\|Ay - bt\|_1 = \frac{\|Ax - b\|_1}{c^T x + d}.$$

The division is made possible because we also assumed $d > \|c\|_1$.

□

Boyd 4.21. (1) Define $v := A^{1/2} x$ so that $x^T A x \leqslant 1$ becomes $\|v\|^2 \leqslant 1$. Then we are trying to minimize $c^T (A^{1/2})^T v$. Define $u^T := c^T (A^{1/2})^T$. The minimizer is $-u/\|u\|_2$. Translating this back to original variables,

$$v^* = -\frac{u}{\|u\|_2} = -\frac{A^{-1/2} c}{\|A^{-1/2} c\|} \implies x^* = A^{-1/2} v = -\frac{A^{-1} c}{\|A^{-1/2} c\|} = -\frac{A^{-1} c}{\sqrt{c^T A^{-1} c}}.$$

If A is not PD, diagonalize it as $A = QDQ^T$. Then $x^T A x \leqslant 1$ becomes $x^T QDQ^T x \leqslant 1$, or $\|D^{1/2} Q x\| \leqslant 1$, and our objective function is $c^T x = c^T Q^T Q x = (Qc)^T (Qx)$. If all eigenvalues of A are positive then this is identical to the previous case.

If the smallest eigenvalue is negative, the answer is $-\infty$. If the smallest eigenvalue is 0, but the corresponding component of Qc is nonzero, then the answer is again unbounded from below. If all components of Qc corresponding to an eigenvalue of 0 are zero, then we reduce the problem into a smaller case with positive eigehvalues, and the result follows from the first case as well.

(2) This is identical to the first problem after a change of variable, yielding

$$x^* = x_c - \frac{A^{-1}c}{\sqrt{c^T A^{-1} c}}.$$

(3) If $B \geq 0$ then the minimum is 0 with $x = 0$. Otherwise, we define $y := A^{1/2}x$ and $C := (A^{-1/2})^T B A^{-1/2}$. Then our objective function becomes $y^T C y$ and the constraint becomes $\|y\| \leqslant 1$. Therefore, the optimal value is given by

$$p^* = \min\left\{0, \min_{\|x\|=1} x^T (A^{-1/2})^T B A^{-1/2} x\right\}.$$

Since any arbitrary vector is a linear combination of eigenvalues, the second quantity is uniquely minimized when y is the smallest eigenvalue of C. □