PHIL 236 Paper #1

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Are regions made up of points, or is space 'gunky' in the sense that there are regions but no point?

- *What sort of primitives would you need to theorize about this kind of space?*
- *Physical theories often make reference to points — a field, for instance, is treated as an assignment of quantities to points. Could these theories be reformulated if space were gunky?*
- *What would it mean for an object to move continuously if space were gunky?*

Introduction

The nature of space has been a fundamental question in philosophy, physics, among other fields, with a long-standing debate between different perspectives: whether the space consists of infinitesimal points or a collection of "gunky" regions without pointlike constituents[1,2]. This paper aims to (i) explore the concept of pointless geometry, in particular adopting "Tarskian gunky geometry[3]", (ii) attempt to reconstruct familiar notions like points, distance, continuity, and so on., and (iii) identify the set of necessary (and/or sufficient) primitives for such notions to hold.

Historically, space is viewed as an infinite collection of infinitesimal points, similar to what one first learns in elementary geometry. This view has been the foundation for much of modern mathematics, including Euclidean geometry, Cartesian system, and calculus. In the recent two centuries, dozens of scholars independently proposed a variety of axioms and systems on pointless geometry. The first of these was by Lobachevsky in 1835, who assumed the primitive notions of *solids* and *contact* between solids. Yet it suffered from obscurity and a lack of rigor, and Lobachevsky did not pursue this subject further[4]. In around 1921 Leśniewski posted a problem regarding a similar framework[5]. In 1929, in his paper *Les fondements de la géometrie des corps*, Tarski proposed a solution to the lingering problems of geometry of solids, using the notions of *spheres* and the *inclusion* between them. My following exposition will base on Tarski's work, as well as PHIL236's lectures on related content.

Much of my inspiration is drawn from topology and real analysis. Do not panic when I decide to introduce ϵ-δ type of argument :')

[1] `https://link.springer.com/chapter/10.1007/978-94-007-0214-1_16`
[2] Since this is a draft I am just going to leave links as-is. A more formal formatting will be used for later versions.
[3] `https://andrew-bacon.github.io/courses/PHIL236/notes#tarskian-gunky-geometry`
[4] `http://www.dipmat2.unisa.it/people/gerla/www/Down/point-free.pdf`
[5] `https://research.vu.nl/ws/portalfiles/portal/807591/1802-002.pdf`

Recovering Euclidean Geometry from Tarskian Primitives

Recall that in lecture we stated that Tarskian geometry began with two primitives: (T1) being in an **open ball**, and (T2) **parthood**, the notion of a ball being contained in another, denoted by "⊂."

Also recall that Euclidean geometry included three primitives: (E1) the notion of a **point**, (E2) **congruence**, and (E3) **betweenness**. Knowing that Tarskian geometry explicitly rejects the existence of points, we shall attempt to establish the equivalence of Tarskian and Euclidean geometry by deriving (E2), (E3) using (T1), (T2).

The definitions go as follows. Let B_1, B_2 be two balls per (T1).

Definition

(1) B_1 and B_2 are **disjoint** if and only if no ball B_3 is contained in both, written $B_1 \cap B_2 = \varnothing$.

(2) Defining tangency:

- B_1 and B_2 are **externally tangent** if:
  - $B_1 \cap B_2 = \varnothing$, and
  - For any two balls that both contain B_1 and disjoint from B_2, one of them must be contained in the other. In other words, for any B_3, B_4 such that $B_1 \subset B_3$, $B_1 \subset B_4$, and $B_3 \cap B_2 = B_4 \cap B_2 = \varnothing$, either $B_3 \subset B_4$ or $B_4 \subset B_3$.
- B_1 and B_2 are **internally tangent** (assuming $B_1 \subset B_2$, resp. $B_2 \subset B_1$) if:
  - $B_1 \subset B_2$, and
  - For any two balls that both contain B_1 and are contained in B_2, one of them must be contained in the other. In other words, for any B_3, B_4 such that $B_1 \subset B_3 \subset B_2$ and $B_1 \subset B_4 \subset B_2$, either $B_3 \subset B_4$ or $B_4 \subset B_3$.

(3) Recovering betweenness (diametrical opposites):

- B_1 and B_2 are at **external diametric opposites** of a ball B_3 if:
  - B_1 and B_3 are externally tangent, and so are B_2 and B_3, and
  - If B_1' contains B_1 but is disjoint from B_3, and likewise for B_2, then $B_1' \cap B_2' = \varnothing$.
- B_1 and B_2 are at **internal diametric opposites** of a ball B_3 if:
  - B_1 and B_3 are internally tangent, and so are B_2 and B_3, and
  - If B_1' is disjoint from both B_1 and B_3, and likewise for B_2, then $B_1' \cap B_2' = \varnothing$.

(4) Defining concentric balls: Assuming $B_1 \subset B_2$, they are **concentric** if, for any two balls B_3, B_4 external diametric opposites of B_1 and (internally) tangent to B, they are also at internal diametric opposites of B_2.

(5) Recovering congruence: For another ball B_3, (B_1, B_3) is **congruent** to (B_2, B_3), written $(B_1, B_3) \equiv (B_2, B_3)$, if there exists a ball B_3' concentric with B_3 such that:

- For all balls B_1' concentric with B_1, either $B_1' \subset B_3'$ or $B_1' \cap B_3' = \varnothing$, and
- Likewise for all balls B_2' concentric with B_2.

Intuitively (by appealing to our common knowledge in point-based geometry), both of our recovered notions (diametrical opposites, congruence) are with respect to the imaginary "center" of the balls. One might be tempted to think that three balls satisfy (3) up to some permutation[6] if their corresponding "centers" are colinear, and that $(B_1, B_3) \equiv (B_2, B_3)$ if their "centers" satisfy the same condition w.r.t. their length. Fortunately, this is indeed a valid thought, thanks to equivalence classes:

Definition: Equivalence Classes

An **equivalence relation** on a set X is a binary operator $\sim$ satisfying:

- (reflexivity) $a \sim a$ for all $a \in X$,
- (symmetry) $a \sim b$ if and only if $b \sim a$, and
- (transitivity) if $a \sim b$ and $b \sim c$ then $a \sim c$.

Given $a \in X$, its **equivalence class** is defined by $[a] := \{x \in X : a \sim x\}$.

Our goal, naturally, is to show that concentricity induces an equivalence relation on $\mathcal{S}$, the collection of all open balls. However, this is nontrivial, and we need a few more primitives[7]:

(T3) Universal ball.[8] The space itself is a "huge" ball; call it the universal ball. Call all other balls "finite balls." Then all finite balls are contained in the universal ball, and no finite ball is tangent to the universal ball.

(T4) Increasing nested balls. Let $\{B_i\}_{i \in \mathcal{I}}$ be a collection of nested, increasing balls indexed over $\mathcal{I}$ (i.e., $B_i \subset B_j$ if $i \leqslant j$). Then $\bigcup_{i \in \mathcal{I}} B_i$ is a well-defined ball. Furthermore, if there exists a ball B, finite or not, such that $B_i \subset B$ for all i, then $\bigcup_{i \in \mathcal{I}} B_i \subset B$.

- Here we implicitly assume that $\mathcal{I}$ is equipped with a (non-strict) total order $\leqslant$. That is, $\leqslant$ is reflexive, transitive, antisymmetric (if $i \leqslant j$ and $j \leqslant i$ then $i = j$), and total (for all i, j, either $i \leqslant j$ or $j \leqslant i$).

(T5) Decreasing nested balls. Let $\{B_i\}_{i \in \mathcal{I}}$ be a collection of nested, decreasing balls indexed over $\mathcal{I}$. Then either $\bigcap_{i \in \mathcal{I}} B_i = \varnothing$ or is a well-defined ball. Furthermore, if there exists a ball B, empty or not, such that $B \subset B_i$ for all i, then $B \subset \bigcap_{i \in \mathcal{I}} B_i$.

(T3) Families of tangent balls. Let B_1, B_2 be externally tangent finite balls. Then there exists an infinite set $\mathrm{Ext}(B_2, B_1)$ of finite balls, each of which is externally tangent to B_1, and for any two balls $B_i, B_j \in \mathrm{Ext}(B_2, B_1)$, either $B_i \subset B_j$ or $B_j \subset B_i$. Further, in this collection, there are balls that contain B_2 as well as balls contained in B_2. Likewise, if B_2, B_1 are internally tangent, then there exists an infinite set $\mathrm{Int}(B_2, B_1)$ of finite balls, each of which is internally tangent to B_1, and for any two ball $B_i, B_j \in \mathrm{Int}(B_2, B_1)$, either $B_i \subset B_j$ or $B_j \subset B_i$. Further, B_2 is neither the largest nor the smallest ball in $\mathrm{Int}(B_2, B_1)$.

(T4) Uniqueness of doubly tangent ball. If $B_1 \subset B_2$ are not internally tangent, then for any $B_3 \subset B_2$ externally tangent to B_1, there exists a unique $B_4 \in \mathrm{Ext}(B_3, B_1)$ externally tangent to B_1 and internally tangent to B_2.

[6] Meaning that if three balls are B_1, B_2, B_3, then for some $\sigma \in S_3$, $\sigma(B_1), \sigma(B_2), \sigma(B_3)$ satisfy (3).
[7] My guess is that there are redundant primitives within the following, but it will take some time before I figure out which ones are redundant.
[8] Cyan definitions are probably not needed. I'll think about this.

Likewise, for any $B_5 \subset B_2$ internally tangent to B_2, there exists a unique $B_6 \in \mathrm{Int}(B_5, B_2)$ internally tangent to B_2 and externally tangent to B_1.

Proof of equivalence relation defined by concentricity. To show reflexivity, let $B_1 = B_2$. If B_3, B_4 are at external diametric opposites of B_1 then in particular they are externally tangent to B_1. This eliminates the possibility of B_3, B_4 being internally tangent to B_2, and (4) holds trivially[9].

To show symmetry, simply swap the order of B_1 and B_2.

Finally, to show transitivity, we assume that $B_1 \subset B_2 \subset B_3$ and B_i and B_{i+1} are concentric. Let B_4, B_5 be at external diametric opposites of B_1 and internally tangent to B_3. Let $B_6 \in \mathrm{Ext}(B_4, B_1)$ be externally tangent to B_1 and internally tangent to B_2, and likewise define $B_7 \in \mathrm{Ext}(B_5, B_1)$. It follows by concentricity of B_1, B_2 that B_6, B_7 are at internal diametric opposites of B_2. Since B_6, B_2 are internally tangent, we can define B_8 to be externally tangent to both. Further, because of $\mathrm{Ext}(B_8, B_2)$ we may assume B_8 is externally tangent to B_2 while also internally tangent to B_3. Similarly define B_9. Then by concentricity of B_2, B_3, we know B_8, B_9 are at internal diametric opposites of B_3. Now (T4) implies that B_4 perfectly contains B_6 and B_8, in the sense that B_6, B_8 are both internally tangent to B_4, while B_4, B_6 are externally tangent to B_1 and B_4, B_8 internally tangent to B_3. This shows that B_4, B_5 are at internal diametric opposites of B_3, and the proof is complete. $\square$

With the equivalence relation proof finally out of the way, we may now safely partition the space of open balls into equivalence classes. And we define the **center** of the concentric balls to be the representative of this equivalence class. We abuse our notation here: for a ball B, denote the set of all balls concentric to it by $\mathcal{C}(B)$.

Time, Motion, and Continuity

Our next goal would be to introduce necessary notions of limits in order to define motion, and in particular, continuous motions. We will assume the substantivalism primitives of time, namely congruence and betweenness between times. For the remainder of this write-up, let $\mathcal{T}$ denote the collection of times. For any two times t_1 and t_2, denote corresponding time interval by (t_1, t_2)[10]. We write $\mathrm{Bet}(x, y, z)$ if time y is between x and z, and we write $(x, y) \equiv (z, w)$ if these two time intervals are congruent. We now define the following.

Definition: Limits in Time

A sequence $\{t_n\}_{n \geqslant 1} \subset \mathcal{T}$ is said to converge **dyadically** to a **limit** $t \in \mathcal{T}$ if for all n, (i) $\mathrm{Bet}(t, t_{n+1}, t_n)$ and (ii) $(t, t_{n+1}) \equiv (t_{n+1}, t_n)$.

Let t_1, t_1', and t be such that $\mathrm{Bet}(t_1, t, t_1')$ and $(t_1, t) \equiv (t, t_1)$. Let $\{t_n\}, \{t_n'\}$ be the corresponding dyadic sequences with limits t. A sequence $\{s_n\}$ is said to **converge** to a **limit** $t \in \mathcal{T}$ if, for all $n \geqslant 1$, there exists a $N \geqslant 1$ such that $\mathrm{Bet}(t_n, s_k, t_n')$ for all $k \geqslant N$. In this case we write $s_n \to t$.

Intuitively, a dyadic sequence converges to a limit by "halving" its distance every time. The general definition of convergence, on the other hand, is a direct analogy to the $\epsilon - \delta$ convergence criterion in a metric space, in the sense treat the tail of the sequence $\{s_n\}$ is uniformly bounded by a certain distance from the limit.

[9]Here we used the fact that $(B_1 \cap B_2 = \varnothing)$ and $(B_1 \subset B_2)$ cannot be both true, yet they are the necessary conditions for being externally/internally tangent.

[10]Not necessarily in this order. It is possible that t_2 happens before t_1.

[10]This means $N(n)$ depends on n.

While this definition intuitively makes sense, there is one caveat: we need to show that dyadic limits are unique. Fortunately this is a one-liner: give any t_n and two distinct candidates s, t, if $s \neq t$ then no t_{n+1} satisfy both conditions for dyadic convergence.

We also need a similar notion in $\mathcal{S}$, the space of balls. The intuition remains: a sequence of balls converges if the tail balls become sufficiently similar. To put that into formal language,

Definition: Limits of Balls

Let $B \in \mathcal{B}$ be a ball and recall that $\mathcal{C}(B)$ denotes the collection of balls concentric to B. A sequence $\{B_n\}_{n \geqslant 1} \subset \mathcal{B}$ is said to **converge** to the **limit** B, written $B_n \to B$, if for all $B', B'' \in \mathcal{C}(B)$ with $B' \subsetneq B \subsetneq B''$, there exists a $N \geqslant 1$ such that $B' \subsetneq B_k \subsetneq B''$ for all $k \geqslant N$.

Intuitively, by considering two balls concentric to the limit B, the definition above requires that the boundaries[11] of the tail of $\{B_n\}$ is uniformly contained in the ring/annulus formed by these two concentric balls. And since the choosing of these two balls is arbitrary, we establish the notion of convergence.
Of course, we can also extend the definitions to when the balls "converge" to the entire space (e.g. a sequence of ever expanding concentric balls), or when the balls "shrink" to nothing, or the "center." The former would drop B'' and the latter would drop B', but that is not of our main interest.

With all important notions defined, we are now ready to cook up the definition of continuous motion, with respect to our space of balls. First, a **motion** is a mapping $\psi : \mathcal{T} \to \mathcal{B}$. If $\psi(t) = B$, we say that at time t, the motion is represented by ball B. We define a motion to be **continuous** if for any converging time sequence $\{t_n\} \to t$, we also have $\{\psi(t_n)\} \to \psi(t)$.

Wrapping Up

So far, in this paper we have attempted to recover important notions in Euclidean geometry by starting with balls in Tarski geometry. We have first recovered the notions of *betweenness* and *congruence*, and with a bit of extra work and introducing additional primitives, we provided an alternate perspective, viewing *points* as equivalence classes of "concentric balls." We also briefly mentioned the notion of convergence with respect to balls and provided a dry definition of continuous motion in this regard.
Unfortunately due to the extremely limited amount of time, I was unable to draw figures like I promised in the previous draft. I would be happy to supplement them in-person. And there are also so many interesting properties of these "balls" which I have not touched upon — I may get into some of those in the next paper.

[11] This is not yet a well-defined term, but intuitively we know what it means.