MATH 507b Assignment 1

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Problem 1

Let $f:[0,1]\to\mathbb{R}$ be continuous. Show the limit

$$\lim_{n\to\infty}\int_{[0,1]^n} f((\prod_{i=1}^n x_i)^{1/n})\,\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_n$$

exists and find it explicitly in terms of f. *Hint: LLN.*

Solution. Let $Y_1,Y_2,\cdots,Y_n$ be i.i.d. exponential, i.e., each with PDF $e^{-y}\mathbf{1}\cdot[x\geqslant 0]$. Write $S_n:=\sum Y_i$. By SLLN, since $\mathbb{E}Y_i=1$, $S_n/n\to 1$ almost surely. Using change of variables $x_i\leftarrow e^{-y_i}$, we have $(\prod_{i=1}^n x_i)^{1/n}=\exp(-S_n/n)$, and consequently

$$\int_{[0,1]^n} f((\prod_{i-1}^n x_i)^{1/n})\,\mathrm{d}x_1\,\mathrm{d}x_2\cdots\mathrm{d}x_n=\int_{[0,\infty)^n} f(\exp(-S_n/n))\,\mathrm{d}\mathbb{P}=\mathbb{E}[f(\exp(-S_n/n))].$$

By SLLN on S_n/n, continuity of f, and Lebesgue DCT, $\mathbb{E}[f(\exp(-S_n/n))]\to\mathbb{E}[f(e^{-1})]=e^{-1}$.

Problem 2

(1) A geometric random variable X has distribution $\mathbb{P}(X=k)=p(1-p)^k$ for $k\geqslant 0$ and $p\in(0,1)$. Let $p_n\in(0,1)$ with $p_n\to 0^+$ and let X_n be geometrically distributed with parameters p_n. Find the weak limit p_nX_n as $n\to\infty$.

(2) Let $p_n\in(0,1)$ be a sequence such that $p_n\to 0$ as $n_0\to\infty$ while $p_n\cdot n\to\lambda\in(0,\infty)$. Let X_n be binomially distributed with parameters p_n. Show that X_n converge weakly and find their limit.

Solution. (1) Since $\mathbb{P}(X_n>x)=(1-p)^x$ (first x throws are all tails),

$$\mathbb{P}(p_nX_n>x)=\mathbb{P}(X_n>x/p_n)=\mathbb{P}(X_n>\lfloor x/p_n\rfloor)=(1-p)^{\lfloor x/p_n\rfloor}.$$

Taking limits,

$$\log\left((1-p)^{\lfloor x/p_n\rfloor}\right)=\lfloor x/p_n\rfloor\log(1-p)\sim(x/p)(-p)=-x$$

where $\sim$ means the limit ratio converges to 1. Therefore $\mathbb{P}(p_nX_n>x)\to e^{-x}$, an exponential distribution with parameter 1.

(2) Here we use the continuity theorem (Durrett 3.3.17),s showing that the ch.f. of X_n converges pointwise to that of a Poisson with parameter λ and Poisson(λ)'s ch.f. is continuous around the origin. The ch.f. of

$X_n \sim \text{Binomial}(n, p_n)$ is

$$\varphi_{X_n}(t) = (p_n \exp(it) + (1 - p_n))^n.$$

Using $p_n \to 0, np_n \to \lambda$, and expanding $\log(1+x)$ as $x \to 0$, we obtain

$$\begin{aligned}\varphi_{X_n}(t) &= \exp(n \log(p_n e^{it} + 1 - p_n)) = \exp(n \log(1 + p_n(e^{it} - 1))) \\ &\to \exp(np_n(e^{it} - 1)) \to \exp(\lambda(e^{it} - 1))\end{aligned}$$

which is the ch.f. of Poisson(λ). It is also clear that this limit is continuous at 0, so we are done.

Problem 3

Skipped.

Problem 4

Suppose X is a non-empty set and $\mathcal{A}$ is a σ-algebra of subsets of X. Show that A is either uncountable or is finite and with cardinality 2^n for some n.

Proof. We first show the finite case. Here, for each $x \in X$ there exists a *smallest* set $S_x \in \mathcal{A}$ containing x. Assuming $\mathcal{A}$ is finite, and noticing that for any $x \neq y$, either $S_x = S_y$ or $S_x \cap S_y = \varnothing$, we see that there exists a finite partition of X using, say, n sets of form S_x. It follows that any set in $\mathcal{A}$ must be a union of sets of form S_x as well. In particular, this shows that $|\mathcal{A}| = 2^n$.

Now assume $\mathcal{A}$ is infinite. Let $\{E_n\}$ be an infinite collection of sets in $\mathcal{A}$ and iteratively define

$$F_1 = E_1 \qquad F_n = E_n \backslash \Big(\bigcup_{i=1}^{n-1} E_i \Big)$$

so that $\{F_n\}$ is pairwise disjoint. But then $\mathcal{A}$ contains an infinite sequence of disjoint sets, so its cardinality is at least $2^{\mathbb{N}}$, uncountable. This completes the proof. $\square$