MATH 507b Homework 2

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Problem 1

(1) (D4.1.3) Suppose that $X, Y \in L^2$ are defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and suppose $\mathcal{G}$ is a σ-algebra contained in $\mathcal{F}$. Imitate the proof in the remark after Theorem 1.5.2 to prove the conditional Cauchy-Schwarz inequality:

$$\mathbb{E}[XY|\mathcal{G}]^2 \leqslant \mathbb{E}[X^2|\mathcal{G}] \cdot \mathbb{E}[Y^2|\mathcal{G}].$$

As part of the problem show that all conditional expectations are well-defined.

(2) (D4.1.7) Let $X \in L^2$ be defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and suppose $\mathcal{G}$ is a σ-algebra contained in $\mathcal{F}$. Let $\mathrm{var}(X|\mathcal{G}) = \mathbb{E}[X^2|\mathcal{G}] - \mathbb{E}[X|\mathcal{G}]^2$. Show that

$$\mathrm{var}(X) = \mathbb{E}[\mathrm{var}(X|\mathcal{G})] + \mathrm{var}(\mathbb{E}[X|\mathcal{G}]).$$

As part of the problem show that all conditional expectations are well-defined.

Proof. (1) By linearity of conditional expectations, for any θ,

$$0 \leqslant \mathbb{E}[(X+\theta Y)^2|\mathcal{G}] = \theta^2\mathbb{E}[X^2|\mathcal{G}] + 2\theta\mathbb{E}[XY|\mathcal{G}] + \mathbb{E}[Y^2|\mathcal{G}].$$

This means the quadratic determinant

$$(2\mathbb{E}[XY|\mathcal{G}])^2 - 4\cdot\mathbb{E}[X^2|\mathcal{G}]\mathbb{E}[Y^2|\mathcal{G}] = 4\mathbb{E}[XY|\mathcal{G}]^2 - 4\mathbb{E}[X^2|\mathcal{G}]\cdot\mathbb{E}[Y^2|\mathcal{G}]$$

must be nonpositive. From this we recover Cauchy-Schwarz for conditional expectations.

On a side note, $\mathbb{E}[X^2|\mathcal{G}]$ and $\mathbb{E}[Y^2|\mathcal{G}]$ are well-defined since $X, Y \in L^2$, and $\mathbb{E}[XY|\mathcal{G}]^2$ is well-defined since Hölder's inequality implies that the product of two L^2 variables is L^1.

(2)

$$\begin{aligned}\mathbb{E}[\mathrm{var}(X|\mathcal{G})] + \mathrm{var}(\mathbb{E}[X|\mathcal{G}]) &= \mathbb{E}\Big(\mathbb{E}[X^2|\mathcal{G}] - (\mathbb{E}[X|\mathcal{G}])^2\Big) + \mathbb{E}((\mathbb{E}[X|\mathcal{G}])^2) - (\mathbb{E}[\mathbb{E}[X|\mathcal{G}]])^2 \\ &= \mathbb{E}X^2 - (\mathbb{E}[X|\mathcal{G}])^2 + \mathbb{E}(\mathbb{E}[X^2|\mathcal{G}]) - (\mathbb{E}X)^2 = \mathrm{var}(X).\end{aligned}$$

□

Problem 2

(1) (D4.3.3) Let X_n, Y_n be positive integrable and adapted to $\mathcal{F}_n$. Suppose that

$$\mathbb{E}[X_{n+1}|\mathcal{F}_n] \leqslant X_n + Y_n,$$

with $\sum Y_n < \infty$ almost surely. Prove that X_n converges almost surely to a finite limit by finding a closely related supermartingale to which Thm 4.1.12 can be applied.

(2) (D4.3.13) Suppose each family has exactly three children but coin flips determine their sex. In the 1800s, only male children kept the family name, so following the male offspring leads to a branching process with $p_0 = 1/8, p_1 = 3/8, p_2 = 3/8$, and $p_3 = 1/8$. Compute the probability ρ that the family name will die out when $Z_0 = 1$.

Solution. (1) Consider $W_n = X_n - \sum_{k=1}^{n-1} Y_k$. It follows that W_n is integrable and $W_n \in \mathcal{F}_n$ as it is the finite sum of $\mathcal{F}_n$-measurable variables. Further, $\{W_n\}$ forms a supermartingale as

$$\mathbb{E}[W_{n+1}|\mathcal{F}_n] = \mathbb{E}[X_{n+1} - \sum_{k=1}^{n} Y_k|\mathcal{F}_n] = \mathbb{E}[X_{n+1}|\mathcal{F}_n] - \sum_{k=1}^{n} Y_k \leqslant X_n + Y_n - \sum_{k=1}^{n} Y_k = W_n.$$

Consider a large M. Define stopping time $N = N_M = \inf\{c : \sum_{k\leqslant c} Y_k > M\}$. It follows that $W_{N\wedge n}$ is a supermartingale. To invoke the supermartingale convergence theorem, we add M to each $W_{N\wedge n}$ observe that the nonnegative variables $W_{N\wedge n} + M$ converge a.s. to an L^1 limit. Hence $\lim_n W_{N\wedge n}$ exists a.s. Finally, since $\sum Y_n < \infty$ a.s., letting $M \uparrow \infty$ we see $\{N_M = \infty\} \uparrow \Omega$ a.s. Therefore $\lim_n W_n$ also exists a.s., and $X_n = W_n + \sum_{k=1}^{n-1} Y_k \to \lim_n W_n + \sum Y_k$ which is finite a.s.

(2) ρ is the root of $\varphi(p) = p_0 + p_1\rho + p_2\rho^2 + p_3\rho^3$ in $[0, 1)$. This gives $\rho = \sqrt{5} - 2$.

Problem 4

Suppose that $\{\mathcal{F}_n\}$ is a filtration of some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let T be a stopping time w.r.t. that filtration.

(1) Define $\mathcal{F}_T = \{A \in \mathcal{F} : A \cap \{T \leqslant n\} \in \mathcal{F}_n \text{ for all } n \geqslant 0\}$. Prove that $\mathcal{F}_T$ is a σ-algebra and T is measurable with respect to it.

(2) If S, T are stopping times, show that $S + T, \max(S, T)$, and $\min(S, T)$ are also stopping times.

(3) Prove that $\mathcal{F}_{\min(S,T)} = \mathcal{F}_T \cap \mathcal{F}_S$.

Proof. (1) Clearly, $\varnothing \in \mathcal{F}_n$ for each n, and that $\{T \leqslant n\} = \bigcup_{k\leqslant n}\{T = k\}$ are also in each $\mathcal{F}_n$ by the definition of stopping time. To show $\mathcal{F}_T$ is closed under complementation, we note that $A^c \cap B = (A \cap B)^c \cap B$:

$$A^c \cap \{T \leqslant n\} = \underbrace{(A \cap \{T \leqslant n\})^c}_{\in \mathcal{F}_n} \cap \underbrace{\{T \leqslant n\}}_{\in \mathcal{F}_n} \in \mathcal{F}_n$$

Finally, for closure under countable unions, if each $A_k \in \mathcal{F}_T$, then for any n,

$$\bigcup_{k\geqslant 1} A_k \cap \{T \leqslant n\} = \bigcup_{k\geqslant 1} \underbrace{(A_k \cap \{T \leqslant n\})}_{\in \mathcal{F}_n} \in \mathcal{F}_n.$$

This completes the proof that $\mathcal{F}_T$ is a σ-algebra.

(2) If S, T are stopping times then

$$\{S + T = n\} = \bigcup_{k=0}^{n} (\underbrace{\{S = k\}}_{\in \mathcal{F}_k \subset \mathcal{F}_n} \cap \underbrace{\{T = n - k\}}_{\in \mathcal{F}_{n-k} \subset \mathcal{F}_n}) \in \mathcal{F}_n.$$

For $\max(S, T)$ and $\min(S, T)$, we note that $\{T \leqslant n\} = \bigcup_{k \leqslant n} \{T = k\}$ where $\{T = k\} \in \mathcal{F}_k \subset \mathcal{F}_n$ for each $k \leq n$, so $\{T \leqslant n\} \in \mathcal{F}_n$. Closure under complementation (taking union with $\{T = n\}$) implies $\{T \geqslant n\} \in \mathcal{F}_n$ as well. Since

$$\begin{aligned}
\{\max(S, T) = n\} &= \{S = n, T \leqslant n\} \cup \{S \leqslant n, T = n\} \\
&= (\{S = n\} \cap \{T \leqslant n\}) \cup (\{S \leqslant n\} \cap \{T = n\})
\end{aligned}$$

$$\begin{aligned}
\{\min(S, T) = n\} &= \{S = n, T \geqslant n\} \cup \{S \geqslant n, T = n\} \\
&= (\{S = n\} \cap \{T \geqslant n\}) \cup (\{S \geqslant n\} \cap \{T = n\})
\end{aligned}$$

and each set of form $\{T = n\}, \{T \leqslant n\}$, and $\{T \geqslant n\}$ are in $\mathcal{F}_n$, we are done.

$\square$