MATH 507b Homework 3

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Problem 1: D4.4.7

Let X_n be a martingale w.r.t. $\mathcal{F}_n = \sigma(X_0, \dots, X_n)$ with $X_0 = 0$ and $\mathbb{E}[X_n^2] < \infty$. If $\lambda > 0$ show that

$$\mathbb{P}\left(\max_{1 \leqslant m \leqslant n} X_m \geqslant \lambda\right) \leqslant \frac{\mathbb{E}[X_n^2]}{\mathbb{E}[X_n^2] + \lambda^2}.$$

Hint: Use the fact that $(X_n + c)^2$ is a submartingale and optimize over c.

Proof. For $c \geqslant -\lambda$, using Doob's inequality on $(X_n + c)^2$, a nonnegative submartingale,

$$\begin{aligned}\mathbb{P}\left(\max_{1 \leqslant m \leqslant n} X_m \geqslant \lambda\right) &= \mathbb{P}\left(\max_{1 \leqslant m \leqslant n} (X_m + c)^2 \geqslant (\lambda + c)^2\right) \\ &\leqslant \frac{1}{(\lambda + c)^2}\mathbb{E}[(X_n + c)^2] \\ &= \frac{1}{(\lambda + c)^2}\mathbb{E}[X_n^2 + c^2 + 2cX_n] \\ &= \frac{\mathbb{E}X_n^2 + c^2}{(\lambda + c)^2}.\end{aligned}$$

To minimize the bound we compute the derivative w.r.t. c and set it to 0:

$$2 \cdot (\lambda + c)^{-3} \cdot (\lambda c - \mathbb{E}X_n^2) = 0 \implies c = \mathbb{E}X_n^2/\lambda.$$

Also, the second derivative at $c = \mathbb{E}X_n^2/\lambda$ is $2\lambda^4/(\lambda + \mathbb{E}X_n^2)^3$ which is positive. Hence $c = \mathbb{E}X_n^2/\lambda$ is the global minimizer, and from this we obtain

$$\mathbb{P}\left(\max_{1 \leqslant m \leqslant n} X_m \geqslant \lambda\right) \leqslant \frac{\mathbb{E}X_n^2 + (\mathbb{E}X_n^2/\lambda)^2}{(\lambda + \mathbb{E}X_n^2/\lambda)^2} = \frac{\mathbb{E}X_n^2 \cdot (1 + \mathbb{E}X_n^2/\lambda^2)}{(\mathbb{E}X_n^2 + \lambda^2)^2/\lambda^2} = \frac{\mathbb{E}X_n^2}{\mathbb{E}X_n^2 + \lambda^2}.$$ □

Problem 2: D4.7.1

Let $X_n, n \leqslant 0$ be a backwards martingale. Suppose that for some $p > 1$ we have $X_0 \in L^p$. Show that there exists a random variable $X_{-\infty}$ such that $X_{-\infty} \in L^p$ and that $X_n \to X_{-\infty}$ both almost surely and in L^p.

Proof. By L^p maximal inequality,

$$\mathbb{E}\left(\sup_{-n \leqslant m \leqslant 0} |X_m|^p\right) \leqslant \left(\frac{p}{p-1}\right)^p \mathbb{E}|X_0|^p.$$

The limiting variable $X_{-\infty}$ exists and $X_n \to X_{-\infty}$ almost surely by Thm 4.7.1. Letting $n \to \infty$ we have $\sup_m |X_m| \in$

L^p, so $X_{-\infty} \in L^p$. Finally, since $|X_n - X_{-\infty}|^p \leqslant (2\sup_m |X_m|^p) < \infty$, by DCT we have

$$\mathbb{E}|X_n - X_{-\infty}|^p \to 0,$$

completing the proof of convergence in L^p. □

Problem 3: D4.8.7

Let S_n be a SSRW starting at 0 and let $T = \inf\{n : S_n \notin (-a, a)\}$ where $a \in \mathbb{N}$.

(1) Compute $\mathbb{E}T$ as a function of a.

(2) Find constants b, c such that $Y_n = S_n^4 - 6nS_n^2 + bn^2 + cn$ is a martingale and use this to compute $\mathbb{E}T^2$ as a function of a.

Solution. (1) By Theorem 4.8.7, $\mathbb{E}T = a^2$.

(2) Observe that $\xi_n^2 = \xi_n^4$ are constant 1. Also observe that ξ_{n+1} is independent of $\mathcal{F}_n$, so $\mathbb{E}[\xi_{n+1}|\mathcal{F}_n] = \mathbb{E}\xi_{n+1} = 0$. This implies

$$\begin{aligned}\mathbb{E}[S_{n+1}^4|\mathcal{F}_n] &= \mathbb{E}[(S_n + \xi_{n+1})^4|\mathcal{F}_n] \\ &= \mathbb{E}[S_n^4 + 4S_n^3\xi_{n+1} + 6S_n^2\xi_{n+1}^2 + 4S_n\xi_{n+1}^3 + \xi_{n+1}^4|\mathcal{F}_n] \\ &= S_n^4 + 6S_n^2 + 1\end{aligned}$$

and similarly

$$\mathbb{E}[S_{n+1}^2|\mathcal{F}_n] = \mathbb{E}[S_n^2 + 2S_n\xi_{n+1} + \xi_{n+1}^2|\mathcal{F}_n] = S_n^2 + 1.$$

Since $\{Y_n\}$ is a martingale, $\mathbb{E}[Y_{n+1}|\mathcal{F}_n] = Y_n$, i.e.,

$$(S_n^4 + 6S_n^2 + 1) - 6(n+1)(S_n^2 + 1) + bn^2 + cn = S_n^4 - 6nS_n^2 + b(n+1)^2 + c(n+1),$$

or

$$(2b - 6)n + (b + c - 5) = 0.$$

This gives $b = 3$ and $c = 2$.

Now since $\mathbb{E}Y_0 = \mathbb{E}Y_{T\wedge n}$,

$$0 = \mathbb{E}[S_{T\wedge n}^4 - 6(T\wedge n)S_{T\wedge n}^2 + 3(T\wedge n)^2 + 2(T\wedge n)].$$

Applying MCT to $\mathbb{E}(T\wedge n)^2$ with $\mathbb{E}T = a^2 < \infty$, as well as DCT to $|S_{T\wedge n}| \leqslant a$, we obtain

$$0 = a^4 - 6a^2\mathbb{E}T + 3\mathbb{E}T^2 + 2\mathbb{E}T = a^4 - 6a^4 + 3\mathbb{E}T^2 + 2a^2$$

so

$$\mathbb{E}T^2 = \frac{5a^4 - 2a^3}{3}.$$