MATH 507b Homework 4

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February 22, 2024

Problem 1: D5.2.1

Suppose that X_n is a Markov chain on the space $(S, \mathcal{G})$ with respect to $\mathcal{F}_n$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $A \in \sigma(X_0, ..., X_n)$ and let $B \in \sigma(X_n, X_{n+1}, ...)$. Use the Markov property to show that for any initial distribution μ we have

$$\mathbb{P}_\mu(A \cap B | X_n) = \mathbb{P}_\mu(A|X_n) \cdot \mathbb{P}_\mu(B|X_n).$$

In words, the past and future are conditionally independent given the present. Hint: write LHS as $\mathbb{E}_\mu[\mathbb{E}_\mu[\mathbf{1}_A \mathbf{1}_B | \mathcal{F}_n] X_n]$.

Proof.

$$\begin{aligned}
\mathbb{P}_\mu(A \cap B | X_n) &= \mathbb{E}_\mu[\mathbb{E}_\mu[\mathbf{1}_A \mathbf{1}_B | \mathcal{F}_n] X_n] && \text{(hint)} \\
&= \mathbb{E}_\mu[\mathbf{1}_A \mathbb{E}_\mu[\mathbf{1}_B | \mathcal{F}_n] | X_n] && (\text{since } \mathbf{1}_A \in \mathcal{F}_n) \\
&= \mathbb{E}_\mu[\mathbf{1}_A \mathbb{E}_\mu[\mathbf{1}_B | X_n] X_n] && \text{(Markov property)} \\
&= \mathbb{E}_\mu[\mathbf{1}_A | X_n] \mathbb{E}_\mu[\mathbf{1}_B | X_n] && (\text{since } \mathbb{E}_\mu[\mathbf{1}_B|X_n] \in \sigma(X_n)) \\
&= \mathbb{P}_\mu(A|X_n) \cdot \mathbb{P}_\mu(B|X_n).
\end{aligned}$$

□

Problem 3: D5.2.6

Suppose that X_n is a Markov chain on $(S, \mathcal{G})$ with respect to $\mathcal{F}_n$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $T_C := \inf\{n \geqslant 1 : X_n \in C\}$. Suppose $S \backslash C$ is finite and for each $x \in S \backslash C$, we have $\mathbb{P}_x(T_C < \infty) > 0$. Show that there exists $N \in \mathbb{N}$, $\epsilon > 0$ such that for all $y \in S \backslash C$ and $k \geqslant 1$,

$$\mathbb{P}_y(T_C > kN) \leqslant (1 - \epsilon)^k. \tag{*}$$

Conclude that for each $y \in S \backslash C$, $\mathbb{E}_y[T_C] < \infty$.

Proof. For each $x \in S \backslash C$ there exist large $n(x) \in \mathbb{N}$ and small $\epsilon(x) > 0$ such that $\mathbb{P}_x(T_C < n(x)) > \epsilon(x) > 0$, since $\mathbb{P}_x(T_C < \infty) > 0$. Since $S \backslash C$ is finite, the following variables are well-defined:

$$n_0 := \max_{x \in S \backslash C} n(x) < \infty \qquad \text{and} \qquad \epsilon_0 := \min_{x \in S \backslash C} \epsilon(x) > 0.$$

It follows immediately that (*) holds for $k = 1$ with n_0 and ϵ_0. For the inductive step, note that (assuming (*)

holds for $k-1$)

$$
\begin{aligned}
\mathbb{P}_y(T_C > kN) &= \mathbb{P}_y(T_C > kN \mid T_C > (k-1)N) \cdot \underbrace{\mathbb{P}_y(T_C > (k-1)N)}_{\leqslant (1-\epsilon)^{k-1}} \\
&\leqslant \mathbb{P}_y(X_n \notin C \text{ for } n \in ((k-1)N, kN] \mid X_n \notin C \text{ for } n \leqslant (k-1)N) \cdot (1-\epsilon)^{k-1} \\
&= \mathbb{P}_y(X_n \notin C \text{ for } n \in ((k-1)N, kN] \mid X_{(k-1)N} \notin C) \cdot (1-\epsilon)^{k-1} \\
&= \sum_{x \in S \setminus C} \mathbb{P}_y(X_n \notin C \text{ for } \ldots \mid X_{(k-1)N} = x) \mathbb{P}_y(X_{(k+1)N} = x) \cdot (1-\epsilon)^{k-1} \\
&= \sum_{x \in S \setminus C} \underbrace{\mathbb{P}_x(T_C > N)}_{\leqslant 1-\epsilon} \mathbb{P}_y(X_{(k-1)N} = x \mid X_{(k-1)N} \notin C) \cdot (1-\epsilon)^{k-1} \\
&\leqslant (1-\epsilon)(1-\epsilon)^{k-1} = (1-\epsilon)^k,
\end{aligned}
$$

which completes the proof of (*). Then it becomes immediately clear that $\mathbb{E}_y[T_C] < \infty$, since the integral of $\mathbb{P}_y(T_C > t)\mathrm{d}t$ is bounded from above by the piecewise constant function taking constant value $\mathbb{P}_y(T_C > kn)$ on $(kN, (k+1)N]$ for each k, and the latter is summable since $\sum_k (1-\epsilon)^k < \infty$. $\square$