MATH 507b Homework 6

Qilin Ye

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Throughout this assignment, I will use $A \approx B$ to denote the relation that $\mathbb{P}(A \Delta B) = 0$.

Problem 1: D6.1.1

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and φ a measure-preserving map. Let $\mathcal{I}$ be the collection of sets $A \in \mathcal{F}$ such that $\mathbb{P}(\varphi^{-1} A \Delta A) = 0$.

(1) Prove that $\mathcal{I}$ is a σ-algebra.

(2) Prove that X is $\mathcal{I}$-measurable if and only if $X \circ \varphi = X$ almost surely.

Proof. (1) Clearly $\varnothing \in \mathcal{I}$, and $\Omega \in \mathcal{I}$ because $\varphi^{-1}\Omega$ has probability 1 by measure preservation. To show that $\mathcal{I}$ is closed under complementation, if $A \in \mathcal{I}$ then $\varphi^{-1} A \approx A$, and so

$$\varphi^{-1}(A^c) = \varphi^{-1}(\Omega \backslash A) = \varphi^{-1}\Omega \backslash \varphi^{-1} A \approx \Omega \backslash \varphi^{-1} A \approx \Omega \backslash A = A^c.$$

Finally, to show that $\mathcal{I}$ is closed under countable union, let $A_i \in \mathcal{I}$. It follows that

$$\varphi^{-1}\Big(\bigcup_i A_i\Big) = \bigcup_i \varphi^{-1}(A_i) \approx \bigcup_i A_i.$$

(2) It suffices to show that $\{X > c\}$ is measurable for any c. Suppose X is $\mathcal{I}$-measurable. For any c,

$$\{X \circ \varphi > c\} = \varphi^{-1}\{X > c\} \approx \{X > c\}.$$

This implies that $X \circ \varphi$ and X have a.s. identical level sets, which implies that they equal almost surely.

Conversely, suppose $X \circ \varphi = X$ almost surely. Then,

$$\{X > c\} \approx \{X \circ \varphi > c\} = \varphi^{-1}\{X > c\}$$

so $\{X > c\} \in \mathcal{I}$, completing the proof. □

Problem 2: D6.1.2

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and φ a measure-preserving map.

(1) Some authors call a set $A \in \mathcal{F}$ *almost invariant* if $\mathbb{P}(A \Delta \varphi^{-1} A) = 0$ and call C *invariant in the strict sense*

if $C = \varphi^{-1}(C)$. Let A be any set, and $B = \bigcup_{n=0}^{\infty} \varphi^{-n}(A)$. Show that $\varphi^{-1}(B) \subset B$.

(2) Let B be any set with $\varphi^{-1}(B) \subset B$ and let $C = \bigcap_{n=0}^{\infty} \varphi^{-n}(B)$. Show that $\varphi^{-1}(C) = C$.

(3) Show that A is almost invariant if and only if there is an invariant (in the strict sense) C such that $\mathbb{P}(A\Delta C) = 0$.

Proof. (1) $\varphi^{-1}(B) = \varphi^{-1}(\bigcup_{n=0}^{\infty} \varphi^{-n}(A)) = \bigcup_{n=1}^{\infty} \varphi^{-n}(A) \subset B$.

(2)

$$\begin{aligned}\varphi^{-1}(C) = \varphi^{-1}(\bigcap_{n=0}^{\infty} \varphi^{-n}(B)) &= \bigcap_{n=1}^{\infty} \varphi^{-n}(B) \\ &\overset{*}{=} \bigcap_{n=1}^{\infty} (\varphi^{-n}(B) \cap B) = B \cap \bigcap_{n=1}^{\infty} \varphi^{-n}(B) = \bigcap_{n=0}^{\infty} \varphi^{-n}(B) = C.\end{aligned}$$

where $(*)$ is because $\varphi^{-n}(B) \subset \varphi^{-n+1}(B) \subset \cdots \subset B$, which implies $\varphi^{-n}(B) \cap B = \varphi^{-n}(B)$.

(3) First suppose A is almost invariant. Define B as in (1) using A, and C as in (2) using B in (1). Then C is strictly invariant, and

$$\begin{aligned}C = \bigcap_{i=0}^{\infty} \varphi^{-i} \bigcup_{j=0}^{\infty} \varphi^{-j}(A) &= \bigcap_{i=0}^{\infty} \bigcup_{j=0}^{\infty} \varphi^{-(i+j)}(A) \\ &= \bigcap_{i=0}^{\infty} \bigcup_{j \geqslant i} \varphi^{-j}(A) = \limsup_{j\to\infty} \varphi^{-j}(A).\end{aligned}$$

This shows that $\mathbb{P}(A\Delta C) = 0$, since

$$\mathbb{P}(A\Delta C) \leqslant \mathbb{P}(A \backslash C) + \mathbb{P}(C \backslash A) = \liminf(A \backslash \varphi^{-j}(A)) + \limsup(\varphi^{-j}(A) \backslash A) = 0.$$

Conversely, suppose for some strictly invariant C we have $\mathbb{P}(A\Delta C) = 0$, so that $\varphi^{-1}(C) = C$. It follows that

$$\varphi^{-1}(A \backslash C) = \varphi^{-1}(A) \backslash C \qquad \text{and} \qquad \varphi^{-1}(C \backslash A) = C \backslash \varphi^{-1}(A).$$

Since φ is measure-preserving, both sets above are of null. But then using a chain argument,

$$\varphi^{-1}(A) \backslash A \subset (\varphi^{-1}(A) \backslash C) \cup (C \backslash A)$$

a null set, and similarly $A \backslash \varphi^{-1}(A)$ is a null set. This completes the proof. □

Problem 3: D6.1.5

Let $X_0, X_1, \ldots$ be a stationary sequence. Suppose g is measurable and define $Y_k = g(X_k, X_{k+1}, \ldots)$.

(1) Prove that Y_n is stationary.

(2) Show that if X_n is ergodic then so is Y_n.

Proof. (1) Fix k and n. Since X_n is stationary, the distribution of $(X_0, X_1, \ldots)$ equals that of $(X_k, X_{k+1}, \ldots)$ for any k. Therefore Y_i and Y_{i+k} have the same distribution, and so $(Y_0, \ldots, Y_n)$ has the same distribution as $(Y_k, \ldots, Y_{n+k})$.

(2) Note that $Y_{k+1} = Y_k \circ \varphi$ since

$$g(X_{k+1}, X_{k+2}, \cdots) = g(X_k \circ \varphi, X_{k+1} \circ \varphi, \cdots).$$

It follows that for any Borel set B,

$$\{(Y_1, Y_2, \cdots) \in B\} = \varphi^{-1}\{(Y_0, Y_1, \cdots) \in B\}.$$

Since X_n is ergodic, $\{(Y_0, Y_1, \cdots) \in B\}$ is invariant if and only if $\mathbb{P}(A) \in \{0, 1\}$. But then the equation above implies that Y_n is also ergodic. □

Problem 4: D6.1.7

Let $\varphi(x) = 1/x - \lfloor 1/x \rfloor$ for $x \in (0,1)$ and $A(x) = \lfloor 1/x \rfloor$. Show that φ preserves $\mu(A) = (\log 2)^{-1} \int_A (1+x)^{-1}\, \mathrm{d}x$.

Proof. Since $\mu([a,b]) = \dfrac{1}{\log 2} \displaystyle\int_a^b \frac{\mathrm{d}x}{1+x} = \frac{1}{\log 2}(\log(1+b) - \log(1+a))$, and so we have

$$\begin{aligned}
\mu(\varphi^{-1}([a,b])) &= \frac{1}{\log 2} \sum_{n=1}^{\infty} \int_{(n+b)^{-1}}^{(n+a)^{-1}} \frac{\mathrm{d}x}{1+x} \\
&= \frac{1}{\log 2} \sum_{n=1}^{\infty} \left[\log \frac{n+a+1}{n+a} - \log \frac{n+b+1}{n+b}\right] \\
&= \frac{1}{\log 2} \lim_{k\to\infty} \sum_{n=1}^{k} [\ldots] \\
&= \frac{1}{\log 2} \lim_{k\to\infty} \left[\log \frac{k+a+1}{k+b+1} + \log(1+b) - \log(1+a)\right] \\
&= \frac{1}{\log 2}(\log(1+b) - \log(1+a)) = \mu([a,b]).
\end{aligned}$$

□