MATH 507b Homework 8

Qilin Ye

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Problem 3: D7.2.3

Fix $0 < a < b < \infty$ and let B_t be a standard Brownian motion. Let $S(\omega)$ denote the set of local maxima of B_t in (a, b). Prove that $S(\omega)$ is a countable dense subset of (a, b) almost surely.

Proof. Fix $c \in (a, b)$. Theorem 7.2.5 implies that with probability 1, $\inf\{s > c : B_t = B_c\} = c$. In particular, with probability 1, there exists a decreasing sequence $s_n \downarrow c$ where with probability one, $B_{s_n} = B_c$.
On the other hand, Theorem 7.2.4 implies that with probability 1, $\inf\{t \geqslant c : B_t > B_c\} = c$. This means that there exists another decreasing sequence $t_n \downarrow c$ where with probability one, $B_{t_n} > B_a$.
We may WLOG assume that all $s_n, t_n \in (c, b)$, and since they both converge downward to s, we may further assume that they are alternating, i.e., $s_1 > t_1 > s_2 > t_2 > \cdots \downarrow c$, with $B_{s_i} = B_c$ and $B_{t_i} > B_c$.
Brownian motions are continuous, so $B_{s_i} = B_{s_{i+1}} = B_c$ along with $B_{t_i} > B_c$ for some $t_i \in (s_i, s_{i+1})$ imply that B_t attains at least one local maxima in (s_i, s_{i+1}). In particular we have identified a sequence of local maxima converging to c from above, so there are local maxima arbitrarily close to c. Since c is arbitrary the proof is complete: given any point in (a, b) and any arbitrary neighborhood around it, we can find local maxima of B_t in it. □

Problem: D7.3.1, 7.3.3

Let $\mathcal{F}_t$ be right continuous.

(1) Let T_n be a sequence of stopping times with respect to $\mathcal{F}_t$. Show that $\inf T_n, \sup T_n, \limsup T_n$, and $\liminf T_n$ are stopping times.

(2) Suppose that B_t is an $\mathcal{F}_t$-adapted Brownian motion. Let F be an F_σ subset of $\mathbb{R}$. Show that $T = \inf\{t \geqslant 0 : B_t \in F\}$ is a stopping time.

Proof. (1) We first note that if S, T are stopping times, then so are $S \vee T$ and $S \wedge T$:

$$\{S \wedge T \leqslant t\} = \{S \leqslant t\} \cup \{T \leqslant t\} \in \mathcal{F}_t$$

and likewise for $S \vee T$.

It follows that if we define $S_1 = T_1$ and $S_k = S_{k-1} \wedge T_k$ iteratively, then as $k \to \infty$, $S_k \downarrow \inf T_n$. Knowing S_k are stopping times, Theorem 7.2.2 implies $\inf T_n$ also is. Likewise we can define $S'_1 = T_1$ and $S'_k = S'_{k-1} \vee T_k$, from which we see $S'_k \uparrow \sup T_n$ and from Theorem 7.2.3 we conclude $\sup T_n$ is also a stopping time.

Finally, $\limsup T_n = \inf(\sup T_n)$ and $\liminf T_n = \sup(\inf T_n)$ with the appropriate double indices, and so they are also stopping times given $\sup T_n, \inf T_n$ are, when taken over an arbitrary collection.

(2) Write $A = \bigcup_{i \geqslant 1} K_i$ where K_i are closed sets. Define $A_n = \bigcup_{i=1}^n K_i$ and $T_n = \inf\{t \geqslant 0 : B_t \in A_n\}$. These are stopping times by Theorem 7.3.4. Since

$$\{t \geqslant 0 : B_t \in A\} = \bigcup_{n \geqslant 1} \{t \geqslant 0 : B_t \in A_n\}$$

we see $T = \inf T_n$ which, by (1), is a stopping time. □