Math 507b Homework 9

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Problem 2: D7.4.2

Let B_t be a standard Brownian motion defined on a probability space $(\mathbb{P}, \mathcal{F}, \Omega)$. Use (7.2.3) to show that $R = \inf\{t > 1 : B_t = 0\}$ is a random variable with pdf

$$f_R(x+1) = \frac{1}{\pi\sqrt{x}(1+x)}\mathbf{1}[x > 0].$$

Proof. (7.2.3), Fubini, and $\mathbb{P}_y(T_0 = t) = \mathbb{P}_0(T_y = t)$ imply that

$$\begin{aligned}\mathbb{P}_0(R \leqslant x+1) &= \int_{\mathbb{R}} p_1(0,y)\mathbb{P}_y(T_0 \leqslant x)\,\mathrm{d}y \\ &= 2\int_0^\infty p_1(0,y)\int_0^x \mathbb{P}_y(T_0 = s)\,\mathrm{d}s\,\mathrm{d}y \\ &= 2\int_0^t\int_0^\infty p_1(0,y)\mathbb{P}_y(T_0 = s)\,\mathrm{d}y\,\mathrm{d}s \\ &= 2\int_0^t\int_0^\infty p_1(0,y)\mathbb{P}_0(T_y = s)\,\mathrm{d}y\,\mathrm{d}s.\end{aligned}$$

Now we differentiate and use (7.4.6):

$$\begin{aligned}f_R(x+1) &= 2\int_0^\infty p_1(0,y)\mathbb{P}_0(T_y = x)\,\mathrm{d}y \\ &= 2\int_0^\infty (2\pi)^{-1/2}\exp(-y^2/2)(2\pi x^3)^{-1/2} y\exp(-y^2/2x)\,\mathrm{d}y \\ &= \frac{1}{\pi x^{3/2}}\int_0^\infty y\exp(-y^2(1+x)/2x)\,\mathrm{d}y = \frac{1}{\pi x^{3/2}}\frac{x}{1+x}.\end{aligned}$$

□

Problem 3: D7.5.2

Let B_t be a standard Brownian motion. Fix $a, b > 0$. Define $X_t = B_t - bt$ and set $\tau = \inf\{t \geqslant 0 : B_t = a + bt\} = \inf\{t \geqslant 0 : X_t = a\}$.

(1) Show τ is a stopping time.

(2) Use the martingale $Y_t = \exp(\theta B_t - \theta^2 t/2)$ with $\theta = b + (b^2 + 2\lambda)^{1/2}$ to show that

$$\mathbb{E}_0[\exp(-\lambda\tau)] = \exp(-a[b + (b^2 + 2\lambda)^{1/2}]).$$

(3) Prove that $\mathbb{P}_0(\tau < \infty) = \exp(-2ab)$.

Proof. (1) Since bt is deterministic, clearly $X_t \in \mathcal{F}_t$ for each t. Then,

$$\{\tau \leqslant t\} = \{X_s = a \text{ for some } s \in [0,t]\} = \{\inf_{s\in[0,t]} \mathbf{1}[X_s = a] = 1\}.$$

For each s, $\mathbf{1}[X_s = a] \in \mathcal{F}_s$. The infimum, therefore, must be measurable w.r.t. the σ-algebra generated by all previous filtrations, which is $\mathcal{F}_t$.

(2) (1) and Theorem 7.5.6 imply that $\exp(\theta B_{\tau\wedge T} - \theta^2(\tau \wedge T)/2)$ is a martingale, and Theorem 7.5.1 implies that $\mathbb{E}_0 Y_{\tau\wedge T} = \mathbb{E}_0 \exp(\theta B_{\tau\wedge T} - \theta^2(\tau \wedge t)/2) = 1$. Since $B_{\tau\wedge T} \leqslant a + b(T \wedge t)$ by construction, BCT implies that as $t \uparrow \infty$, for $\tau < \infty$,

$$\mathbb{E}_0(\exp(\theta(a + b\tau) - \theta^2\tau/2)) = 1.$$

Now, substituting $\theta = b + (b^2 + 2\lambda)^{1/2}$ into the equation we obtain $\mathbb{E}_0(\exp(\theta a - \lambda\tau)) = 1$. This completes the proof.

(3) Let $\lambda \to 0$. The LHS satisfies $\mathbb{E}_0(\exp(-\lambda r)) \to \mathbb{P}_0(\tau < \infty)$, since $\exp(-\lambda\tau) \to 1$ if $\tau < \infty$ and 0 otherwise. The RHS clearly converges to $\exp(-2ab)$, since

$$\lim_{\lambda\to 0}(b^2 + 2\lambda)^{1/2} = (b^2)^{1/2}.$$ □

Problem 4: D7.5.4

(1) (D7.5.4) Let B_t be a SBM. Let $T = \inf\{t \geqslant 0 : B_t \notin (a,b)\}$ where $a < 0 < b$ and $a \neq -b$. Apply Cauchy-Schwarz to $\mathbb{E}[TB_T^2]$ to conclude that (i) $E[T^2] \leqslant 4\mathbb{E}[B_T^4]$ and (ii) $\mathbb{E}[B_T^4] \leqslant 36\mathbb{E}[T^2]$.

(2) (D8.1.1) If X, U, V and $T_{U,V}$ as in the proof of Thm 8.1.1 prove that $\mathbb{E}[T_{U,V}^2] \leqslant 4\mathbb{E}[X^4]$.

Proof. (1) As in the proof of Theorem 7.5.9 we have

$$\mathbb{E}(B_T^4 - 6TB_T^2) = -3\mathbb{E}T^2.$$

Cauchy-Schwarz on $\mathbb{E}[TB_T^2]$ then gives

$$\mathbb{E}[T^2] \leqslant 2\mathbb{E}[TB_T^2] \leqslant 2(\mathbb{E}T^2)^{1/2}(\mathbb{E}B_T^4)^{1/2}$$

so $\mathbb{E}[T^2] \leqslant 4\mathbb{E}[B_T^4]$. Similarly,

$$\mathbb{E}(6TB_T^2 - B_T^4) = 3\mathbb{E}T^2$$

so

$$\mathbb{E}[B_T^4] \leqslant 6\mathbb{E}[TB_T^2] \leqslant 6(\mathbb{E}T^2)^{1/2}(\mathbb{E}B_T^4)^{1/2},$$

which implies (ii).

(2) From Skorokhod's representation theroem (thm 8.1.1) $B_{T_{U,V}}$ has the same distribution as X. Hence

$$\mathbb{E}[T_{U,V}^2] = \int \mathbb{E}_0[T_{a,b}^2]\,\mathrm{d}\mathbb{P} \leqslant 4\int \mathbb{E}_0[B_{T_{a,b}}^4]\,\mathrm{d}\mathbb{P} = 4\mathbb{E}_0[B_{T_{U,V}}^4] = 4\mathbb{E}[X^4].$$ □