MATH 507B: MIDTERM

SPRING 2024

DUE ON SUNDAY 03.10.2024

You are free to use any textbook as long as you reference it. You are not allowed to collaborate with other students while doing the midterm.

Problem 1. Do the following problems.

a. Let X_n and Y_n be martingales with respect to the same filtration $\mathcal{F}_n$ and $\mathbb{E}[X_n^2] < \infty$, $\mathbb{E}[Y_n^2] < \infty$. Prove that

$$\mathbb{E}[X_n Y_n] - \mathbb{E}[X_0 Y_0] = \sum_{m=1}^{n} \mathbb{E}[(X_m - X_{m-1})(Y_m - Y_{m-1})].$$

b. Let X_n, $n \geq 0$ be a martingale and set $\xi_n = X_n - X_{n-1}$ for $n \geq 1$. Suppose that $\mathbb{E}[X_0^2] < \infty$ and $\sum_{m=1}^{\infty} \mathbb{E}[\xi_m^2] < \infty$. Prove that X_n converges almost surely and in L^2 to some random variable X_∞ as $n \to \infty$.

c. Let X_n, $n \geq 0$ be a martingale with $X_0 = 0$, and set $\xi_n = X_n - X_{n-1}$ for $n \geq 1$. Suppose that $b_m > 0$ is an increasing sequence with $\lim_{m\to\infty} b_m = \infty$. Suppose that $\sum_{m=1}^{\infty} b_m^{-2} \cdot \mathbb{E}[\xi_m^2] < \infty$. Prove that X_n / b_n converges to 0 almost surely and in L^2 as $n \to \infty$.

Problem 2. Let $S_n = \xi_1 + \cdots + \xi_n$ where ξ_i are independent (not necessarily identically distributed) with $\mathbb{E}[\xi_i] = 0$ and $\mathrm{Var}(\xi_i) = \sigma^2 \in (0, \infty)$. Put $\mathcal{F}_n = \sigma(\xi_1, \ldots, \xi_n)$.

a. Prove that $X_n = S_n^2 - \sigma^2 \cdot n$ is an $\mathcal{F}_n$-martingale.

b. Fix $a > 0$ and define $T = \min\{n \geq 0 : |S_n| > a\}$. Show that $\mathbb{E}[T] \geq a^2/\sigma^2$. Hint: It is enough to consider the case $\mathbb{E}[T] < \infty$.

c. Suppose that N is any stopping time with $\mathbb{E}[N] < \infty$. Prove that $\mathbb{E}[S_N^2] = \sigma^2 \cdot \mathbb{E}[N]$.

Problem 3. Suppose that X_n is a Markov chain on a finite or countable space S with transition function p. A function $f : S \to \mathbb{R}$ is said to be superharmonic if $f(x) \geq \sum_{y\in S} p(x,y) f(y)$ or equivalently if $f(X_n)$ is a supermartingale. Suppose that p is irreducible. Show that p is recurrent if and only if every nonnegative superharmonic function is constant.

Problem 4. Suppose that S is a countable state space and p is a transition probability on S. Define

$$\alpha_n = \sup_{i,j\in S} \frac{1}{2} \sum_{k\in S} |p^n(i,k) - p^n(j,k)|.$$

a. Show that $\alpha_{m+n} \leq \alpha_n \cdot \alpha_m$.

b. Show that $\lim_{n\to\infty} \frac{1}{n} \cdot \log \alpha_n = \inf_{m\geq 1} \frac{1}{m} \log \alpha_m$. Hint: use Lemma 2.7.1, D.

c. Show that if $\alpha_m < 1$ for some $m \geq 1$, then we can find constants $A, a > 0$ such that $\alpha_m \leq A \cdot e^{-am}$ for all $m \geq 1$.