[Tentative Title]

# A Mathematical Framework for Harmonies

[Intro]

The foundation of musical composition lies in the blending of individual notes and the progression of chords. When conducting harmonic analysis on a composition, one essentially explores the texture, emotion, and in particular the color created by these notes and chords. From the early days of monophony to polyphony since the late Medieval period, and through the Baroque, Classical, and Romantic eras, the perception of what sounds "pleasant" and what not has evolved continuously. For instance, the interval of the perfect fifth was always considered harmonious, while it took a few extra centuries for the major third and major sixth intervals to become fully embraced by composers (Tenney 1988). Musicians from various time periods have defined their own concepts of "harmony" and "dissonance," yet the tritone, an interval covering three whole steps, was consistently labeled as the "diabolus in musica" or the "devil in music." Throughout history, the tritone has been associated with instability, discordance, and even fear (Helmholtz 2009). Even in later centuries, where composers gained more freedom to experience new harmonic possibilities and tritones became more frequently used, they nevertheless still serve almost only one purpose: to create tension that awaits to be resolved. In this paper, I will examine the historical factors contributing to the tritone's reputation as the "devil interval." Through mathematical analysis, I will propose a mathematical framework that distinguishes harmony from dissonance or, more generally, random combination of two or more voices. Using my definitions, I explain why tritones are often considered discordant and further, how resolutions of suspensions work.

[Defining the Frequency Ratio of Harmonic Intervals]

*In this section I will introduce modern nomenclature, and in particular, the notion of "cents," a logarithmic unit that is used to measure musical intervals. I will also include a brief history of the inevitable issue of irrational numbers when one attempts to represent musical notes via string lengths (Fauvel, Flood, Wilson 2006). Below (anything italicized) are some useful snippets / rough drafts that I will likely include in my formal write-up.*

One of the earliest attempts to quantize musical intervals was by Pythagoras, where he observed that the ratio of frequencies of two notes that form an octave is roughly 2:1. Similarly, the ratio for a perfect fifth is 3:2, perfect fourth 4:3, and whole tone 9:8 (Pythagoras, trans. Barker 1989, 111). In 6th century A.D., Roman philosopher Boethius realized that five whole tones overshoot an octave, and six tones overshoot, but dividing a whole tone in half would lead to a geometric mean in frequencies, inducing an irrational proportion (Boethius 1989). To tackle this issue, Boethius subdivided a whole tone into a "major semitone" and a "minor semitone."

Unfortunately, this technique, known as *Pythagorean Tuning*, has a few serious drawbacks. To implement this method, one iteratively increase or decreases the frequency by a factor of 1.5 to create a complete circle of fifths, and multiply or divide by 2 to adjust for octaves. Some simple mathematical calculation shows that the frequency ratio of an octave is not exactly 2. This means pairs of enharmonic notes may end up with different frequencies at different ends of the circle of fifths. This slight difference, called the Pythagorean comma, is precisely the difference between a major semitone and a minor semitone. Furthermore, while Pythagorean tuning ensures that fifths maintain a perfect 3/2 frequency ratio (and consequently, a perfect 4/3 for fourths), the thirds are represented by a much uglier fraction: 81/64.

Pythagorean Tuning maintained its popularity until the 16th century, when Nicola Vicentino (1511-c.1576) invented *Just Tuning.* Instead of maintain a fixed ratio between certain intervals, Vicentino manually assigned a ratio between the frequency of any note and that of the tonic. He followed Pythagoras' idea and assigned simple fractions to whole tone, thirds, fourths, fifths, and octaves, and additionally set the frequency ratio of a perfect thirds to be 5/4. For other intervals, he assigned fractions based on a series of multiplication of his existing fractions. For example, a major seventh can be broken down into a major third plus a perfect fifth, so he sets the frequency of a major seventh to 5/4 * 3/2 = 15/8.

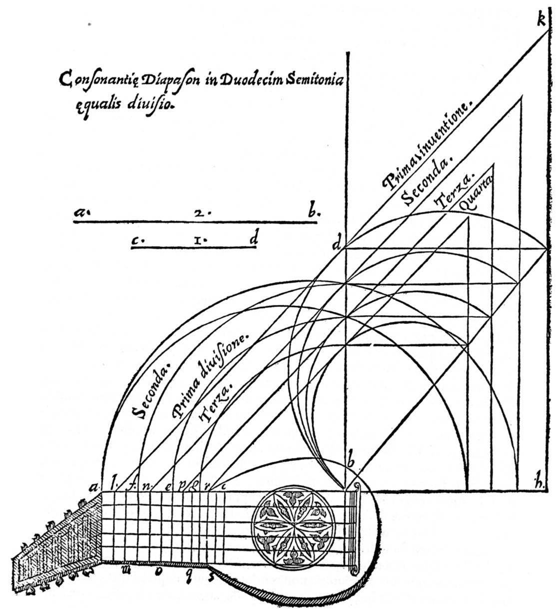
In addition to fixing the issue of a somewhat dissonant thirds, Just Tuning is also invariant under modulation, so a piece of music would maintain its harmony even when played under a different key. Within the same key, however, translations of identical chords may create different effects. For example, setting C as tonic, both C-D and D-E are whole tones, but the frequency ratio between C-D is 9/8 whereas that for D-E is 10/9, leaving a clearly audible 25% difference.

Previous literature has pointed out one similarity between the two tuning systems — both are results of human prejudice. Boethius and Vicentino manipulated the fractions so that what they considered harmonious had a simple fraction, and what they considered dissonant had much more complex ones. Back in the 6th century, thirds and sixths were considered dissonances, which might justify why he only assigned a nice proportion to the fifths. This could very well explain why Vicentino assignsed a stunning 45/32 to a tritone, while this ratio was only 0.4% away from a much simpler 7/5 (Pesic 2010).

[Equal Temperament]

*Introduce Equal Temperament and show how it maintains the advantages of both systems: (i) invariant under modulation, (ii) invariant under translations, and further (iii) is unbiased to human prejudice.*

*To further justify the point that a chord does not need to have a perfect simple fraction to sound pure, quote (Vos, 1988)'s experience that almost all western musicians preferred ET over JT, even though most intervals in ET are actually irrational!!*




*(Image from IMSLP, Zarlino 1588; caption translates to "the equal division of the consonance of a diapason into twelve semitones.")*

[Introducing Math]

- *Historically, it is believed that a consonance happens when the frequency ratio is a simple fraction, i.e., with a small denominator. Intuitive this makes sense: if the sound waves of the two notes share a rather short common cycle, then human ear would perceive them as pleasing and harmonious, because the two notes collectively create a highly periodic sound wave. In contrast, a dissonance occurs when the* *frequency ratio is a more complex fraction, or even worse, irrational. In these cases the resulting sound wave has less regularity and is barely or not periodic at all.*

- *Problem: what is consonant and what is dissonant is subject to human bias — as the two systems have demonstrated, one can manipulate the ratios so that what they consider consonant maintain nice ratios whereas what they consider dissonant have complex ratios.*
- *Solution: introduce a margin of error, say 1%. We then say an interval is "nice" if it is within 1% from a nice fraction x, i.e., [99x/100, 101x/100]. A "nice" fraction may be defined as a fraction with denominator <=5, for example.*
- *Formulate everything into a formal theorem.*

[Compare Definition against ET]

*2^(1/12) is ~0.67% from 16/15*

*2^(2/12) is ~0.22% from 9/8 (whole tone)*

*2^(3/12) is ~0.89% from 6/5 (minor third)*

*2^(4/12) is ~0.79% from 5/4 (major third)*

*2^(5/12) is ~0.11% from 4/3 (major fourth)*

*2^(6/12) is ~1.01% from 7/5 (tritone!)*

*2^(7/12) is ~0.11% from 3/2 (major fifth)*

*2^(8/12) is ~0.78% from 8/5 (minor sixth)*

*2^(9/12) is ~0.90% from 5/3 (major sixth)*

*2^(10/12) is ~0.22% from 16/9 (minor seventh)*

[Analyses]

*Tritone was deemed dissonant because PT's tritone has a ~1.7% error from 7/5. Compared to other intervals in ET, its frequency ratio is still relatively far from any "nice" fractions.*

[Observations & More Points to Expand]

*Q: 1.01% is large, but not too large. Can we make it work?*
*A: yes, consider dominant sevenths, where the the 3rd and 7th form a tritone. Simple reason: the ratio with respect to the lowest note (assuming root inversion), the ratios of the 3rd, 5th, and 7th are 4/3, 3/2, and 16/9. When combined into one chord, the four notes are still representable with a relatively short period.*
*A: observation:* ***I conjecture that the smaller the common denominator of a chord is, the more pleasing and stable it sounds.***
*Q: but why does a consonant chord need resolution?*
*A: consider a major chord on the tonic. A third approximates 5/4 and a fifth approximates 3/2. Together, a root inversion major triad has a period that is even shorter than that of a dominant seventh. Same thing for 1nd inversion: 6/5 for minor third and 8/6 for minor sixth. Same thing for 2nd inversion: 4/3 for fourth and 5/3 for sixths.* ***Potentially consider more chords emerging in classical / romantic period, e.g., variants of augmented sixths, and list concrete musical examples here, for example, Beethoven's Appassionata mvt1, exposition, before running 16th.*** *(… and of course, more citations)*
*Q: is tritone really that bad?*
*A: no, not really. Imagine if you have a magical string, where the left endpoint is C and the high endpoint is C an octave higher. You press any place on the string, and it outputs a sound based on the logarithmic distance to the two endpoints (so it follows ET). (Showing using math — introduce the notions of measure and open cover — that) with almost probability 0.99, you will hear something more dissonant than a tritone. So yes, it is not as consonant as a third, fourth, fifth, or octave, but it really isn't that bad.*

[References]

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More citations to come from musical examples. And maybe anything I think of in the future!