

Some Random Thoughts

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Tarski considered regular open sets. A set A is **regular open** if $A = \text{Int Cl } A$.

Boolean Algebra of RO Sets

Define a **boolean algebra** \mathcal{B} of regular open sets in \mathbb{R}^n as follows:

- join: $A \vee B = \text{Int Cl}(A \cup B)$
- meet: $A \wedge B = A \cap B$
- Boolean complement: $-A = \text{Int Cl}(\mathbb{R}^n \setminus A) = \text{Int}(\mathbb{R}^n \setminus A) = \text{Int}(A^c)$
- $0, 1$ represented by \emptyset and \mathbb{R}^n

More definitions:

- (1) A poset P is a **lattice** if **join** ($x \vee y$) and **meet** ($x \wedge y$) are defined for all x, y . It is **complete** if the joins and meets can be taken over arbitrary sets.
- (2) A lattice A is **distributive** if joins and meets obey

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \quad \text{for all } a, b, c.$$

- (3) $\langle \mathcal{B}, \vee, \wedge, -, \mathbf{0}, \mathbf{1} \rangle$ is a **Boolean algebra** if¹:

- (i) $\langle \mathcal{B}, \vee, \wedge \rangle$ is a distributive lattice,
- (ii) $a \vee \mathbf{0} = a$ and $a \wedge \mathbf{1} = a$ for all a , and
- (iii) $a \vee (-a) = \mathbf{1}$ and $a \wedge (-a) = \mathbf{0}$ for all a .

Proof that \mathcal{B} is boolean algebra. We need to prove that \mathcal{B} is a distributed, complemented lattice. It is clear that $\langle \mathcal{B}, \vee, \wedge \rangle$ form a lattice. (ii) is also clear. To prove (iii), it might be convenient to denote $\text{Int Cl}(A)$ by A^{**} , where $A^* := \mathbb{R}^n \setminus \text{Cl } A = (\text{Cl } A)^c$. For any $A \in \text{RO}(\mathbb{R}^n)$,

$$\begin{aligned} A \vee (-A) &= (A \cup (-A))^{**} = (A \cup A^*)^{**} \\ &= (A^* \cap A^{**})^* = (A^{**} \cap A)^* = \emptyset^* = \mathbf{1} = \mathbb{R}^n \end{aligned}$$

¹Davey and Priestley

and

$$A \wedge (-A) = A \cap \text{Int}(A^c) \subset A \cap A^c = \emptyset.$$

It remains to prove distributivity. First note that

$$A \wedge (B \vee C) = A \wedge (B \cup C)^{**} = A^{**} \cap (B \cup C)^{**}$$

and that

$$(A \wedge B) \cup (A \wedge C) = ((A \wedge B) \cup (A \wedge C))^{**} = ((A \cap B) \cup (A \cap C))^{**} = (A \cap (B \cup C))^{**}.$$

It remains to connect these two chains of equalities — $B \cup C$ is open regular, and

$$X^{**} \cap Y^{**} = X \cap Y = (X \cap Y)^{**} \quad \text{for any } X, Y \in \text{RO}(\mathbb{R}^n).$$

The other distributive identity can be proven analogously. □

Proof that \mathcal{B} is complete. Let I be an index set. Then

$$X = \bigvee_{i \in I} A_i = \text{Int Cl} \bigcup_{i \in I} A_i \quad \text{and} \quad Y = \bigwedge_{i \in I} A_i = \text{Int Cl} \bigcap_{i \in I} A_i$$

are well-defined in $\text{RO}(\mathbb{R}^n)$.

We show that X is the supremum of $\{A_i\}$. For each $i \in I$,

$$A_i = \text{Int Cl} A_i \subset \text{Int Cl} \bigcup_{i \in I} A_i = X.$$

On the other hand if $X' \in \text{RO}(\mathbb{R}^n)$ is another upper bound of $\{A_i\}_{i \in I}$ then $A_i \subset X'$ for all $i \in I$, so $\bigcup A_i \subset X'$, and

$$X = \text{Int Cl} \bigcup_{i \in I} A_i \subset \text{Int Cl} X' = X'.$$

That Y is the infimum can be proven analogously. To avoid abuse of symbolic variables we will denote X, Y by $\sup\{A_i\}_{i \in I}$ and $\inf\{A_i\}_{i \in I}$, respectively. □

Fat Cantor Set Breaks Lebesgue Measure on $\text{RO}(\mathbb{R}^n)$

Consider the Smith-Volterra-Cantor set \mathcal{C} defined by $I = [0, 1]$, and in each iteration define C_k to be the collection of 2^{k-1} subintervals of length 4^{-k} , each of which sits at the middle of an interval from $I \setminus \bigcup_{i=1}^{k-1} C_i$. Let $m(\cdot)$ be the usual Lebesgue measure. The fat Cantor set is $I \setminus \bigcup_{k \geq 1} C_k$. Since C_k consists of 2^{k-1} intervals, each with length 4^{-k} , $m(C_k) = 2^{-k}/2$. Therefore it is clear that $m(\bigcup_{k \geq 1} C_k) = 1/4 + 1/8 + 1/16 + \dots = 1/2$ and so $m(\mathcal{C}) = 1 - 1/2 = 1/2$.

Define *Big Cantor* and *Small Cantor* as in Russell: the union of even-indexed C_i , and the union of odd-indexed C_i , respectively. It is immediately clear that both sets are Lebesgue-measurable, with measures $m(\text{Big Cantor}) = 1/3$ and $m(\text{Small Cantor}) = 1/6$. While their Lebesgue measures behave nicely under usual set-theoretic operations, things look quite different when we turn into Boolean operations defined above. To see this, observe that $\text{Big Cantor} \wedge \text{Small Cantor} = \emptyset$, so $m(\text{Big Cantor} \wedge \text{Small Cantor}) = 0$. On the other hand, since the fat Cantor set \mathcal{C} is totally disconnected, the closure of $I \setminus \mathcal{C} = \bigcup_{k \geq 1} C_k$ is just I , and so $\text{Big Cantor} \vee \text{Small Cantor} = \text{Int } I = I$. Putting everything together, $1 = m(\text{Big Cantor} \vee \text{Small Cantor}) \neq m(\text{Big Cantor}) + m(\text{Small Cantor}) = 1/2$.

Specifically, there are cases where $A \vee B = \text{Int Cl}(A \cup B) \not\supseteq (A \cup B)$, and they lead to unwanted behaviors. In the previous case, every point in $\mathcal{C} = I \setminus (\text{Big Cantor} \cup \text{Small Cantor})$ can be viewed as a limit point of the C_k 's, so $\text{Cl}(\text{Big Cantor} \cup \text{Small Cantor}) = I$, whereas $(\text{Big Cantor} \cup \text{Small Cantor}) \cap \mathcal{C} = \emptyset$.

The pathology arises from the fact that the boundaries of *Big Cantor* and *Small Cantor* have positive measure, where the **boundary** of a set A is defined to be $\partial A = \text{Cl}(A) \setminus \text{Int}(A)$. In our example, the closure operator Cl introduces “too much” to the extent where Lebesgue measure breaks.

To fix Lebesgue measure's finite additivity, one natural solution would be to restrict our attention to a smaller set of algebra: we consider $\text{RC}_0(\mathbb{R}^n)$, the collection of regular open sets whose boundaries have zero Lebesgue measure. This set excludes pathological examples like *Big Cantor* and *Small Cantor*, and indeed preserves finite additivity of Lebesgue measures.

Proof. Suppose $A, B \in \text{RC}_0(\mathbb{R}^n)$ and $A \wedge B = A \cap B = \emptyset$. In particular the interiors are disjoint, so

$$\text{Cl}(A) \cap \text{Cl}(B) = (\text{Int } A \cup \partial A) \cap (\text{Int } B \cup \partial B) = (\partial A \cap \text{Cl } B) \cup (\partial B \cap \text{Cl } A).$$

Since $m(\partial A) = m(\partial B) = 0$ by assumption, the above set has measure 0. Then

$$\begin{aligned} m(\text{Cl}(A \cup B)) &= m(\text{Cl}(A) \cup \text{Cl}(B)) \\ &= m(\text{Cl}(A)) + m(\text{Cl}(B)) - m(\text{Cl}(A) \cap \text{Cl}(B)) \\ &= m(\text{Cl}(A)) + m(\text{Cl}(B)) = m(A) + m(B). \end{aligned}$$

It remains to notice that $\partial(A \cup B) \subset \partial A \cup \partial B$. Thus $A \cup B$ has null boundary and $m(\text{Cl}(A \cup B)) = m(\text{Int Cl}(A \cup B))$, completing the proof. \square