Unfortunately, a point-less Euclidean geometry does not come without technical challenges. One particular complication is the notion of size or measure: without imposing additional structural constraints onto these regular open sets, Lebesgue measure, our most commonly used notion of size, breaks down even on finitely many disjoint sets. To see this, we appeal to the famous **Smith-Volterra-Cantor set**, also known as the Fat Cantor Set. Let m denote the Lebesgue measure.

Define $I = [0,1]$ the unit interval. In the first iteration, remove a subinterval of length $1/4$ from the middle of I, and use I_1 to denote this removed interval: $I_1 = [3/8, 5/8]$. Now $I \backslash I_1$ consists of two disjoint intervals of equal length. In the second iteration, remove two subintervals of length 4^{-2} from each of the two remaining intervals in $I \backslash I_1$, and use I_2 to denote the union of these two removed intervals. Iterative, in the k^{th} iteration we remove intervals of length 4^{-k} from the middle of each of the 2^{k-1} remaining subintervals in $I \backslash \bigcup_{i=1}^{k-1} I_i$ and define I_k accordingly. Since I_k consists of 2^{k-1} disjoint intervals, each with length 4^k, $m(I_k) = 2^{-k}/2$. And clearly $I_i \cap I_j = \varnothing$ for $i \neq j$, so $m(\bigcup_{k \geqslant 1} I_k) = 1/4 + 1/8 + \cdots = 1/2$. We define the Fat Cantor Set to be $\mathcal{C} = I \backslash \bigcup_{k \geqslant 1} I_k$ and it follows that $m(\mathcal{C}) = 1 - 1/2 = 1/2$.

Before visiting pathological examples, it is worth comparing this fat $\mathcal{C}$ against the standard middle-thirds Cantor set, the most notable difference being that $\mathcal{C}$ has a nonzero Lebesgue measure. Despite so, $\mathcal{C}$ is still totally disconenected and is the boundary of $I \backslash \mathcal{C} = \bigcup_{k \geqslant 1} I_k$, just like the standard Cantor set. Given $x \in \mathbb{R}$ and $S \subset \mathbb{R}$, define the point-set distance $d(x, S) = \inf_{s \in S} |x - s|$. It follows that for any $x \in [0,1] = I$, after the first iteration, $d(x, I \backslash I_1) < 1/2$, and by induction $d(x, I \backslash \bigcup_{i=1}^{k} I_i) < 2^{-k}$. This shows that given any $x \in [0,1]$ and $\epsilon > 0$, there exists a sufficiently large k such that $(x - \epsilon, x + \epsilon) \cap I_k \neq \varnothing$. In other words, the closure of $\bigcup_{k \geqslant 1} I_k$ is $[0,1]$.

Now we partition $\bigcup_{k \geqslant 1} I_k$ into two parts, a *Big Cantor* defined by $\bigcup_{k \text{ odd}} I_k$ and a *Small Cantor* = $\bigcup_{k \text{ even}} I_k$. (So far we have *Big Cantor*, *Small Cantor*, and $\mathcal{C}$, whose disjoint union is I.) It is immediately clear that these sets are Lebesgue measurable, with $m($*Big Cantor*$) = 1/3$ and $m($*Small Cantor*$) = 1/6$. While their Lebesgue measures behave nicely under usual set-theoretic operations, things look rather different when we turn into Boolean operations defined above. To see this, observe that *Big Cantor* $\wedge$ *Small Cantor* $= \varnothing$, so $m($*Big Cantor* $\wedge$ *Small Cantor*$) = 0$. On the other hand, *Big Cantor* $\vee$ *Small Cantor* $= \operatorname{Int}\operatorname{Cl}($*Big Cantor* $\cup$ *Small Cantor*$) = \operatorname{Int}\operatorname{Cl}(\bigcup_{k \geqslant 1} I_k) = \operatorname{Int}([0,1]) = [0,1]$, but

$$1 = m(\textit{Big Cantor} \vee \textit{Small Cantor}) \neq m(\textit{Big Cantor}) + m(\textit{Small Cantor}) = 1/2.$$

Here we constructed examples in $\mathbb{R}$. Higher-dimensional counterparts can be constructed analogously.

The problem here is that we constructed two sets A, B such that $A \vee B = \operatorname{Int}\operatorname{Cl}(A \cup B) \supsetneq (A \cup B)$, and this leads to unwanted behaviors. The root of this pathology lies in the fact that both *Big Cantor* and *Small Cantor* (i) have boundaries of positive measure, and (ii) their boundaries are both $\mathcal{C}$. Consequently, "too much" is introduced by the closure operator Cl to the extent where Lebesgue measure breaks. A few modifications have been proposed and explored by previous literature.

The first one is proposed by Lando and Scott, where we simply restrict our attention to sets with boundaries of Lebesgue measure zero, which we denote by $\operatorname{RON}(\mathbb{R}^n) \subset \operatorname{RO}(\mathbb{R}^n)$. (In their paper, they began with regular closed sets instead of regular open sets, but all of the following results hold by taking appropriate complementations and swapping orders of interior and closure operators.) This set explicitly excludes the pathological *Big/Small Cantor* example above, and indeed preserves finite additivity of Lebesgue measures.

Proof. Suppose $A, B \in \operatorname{RC}_0(\mathbb{R}^n)$ and $A \wedge B = A \cap B = \varnothing$. In particular the interiors are disjoint, so

$$\operatorname{Cl}(A) \cap \operatorname{Cl}(B) = (\operatorname{Int} A \cup \partial A) \cap (\operatorname{Int} B \cup \partial B) = (\partial A \cap \operatorname{Cl} B) \cup (\partial B \cap \operatorname{Cl} A).$$

Since $m(\partial A) = m(\partial B) = 0$ by assumption, the above set has measure 0. Then

$$\begin{aligned} m(\mathrm{Cl}(A \cup B)) &= m(\mathrm{Cl}(A) \cup \mathrm{Cl}(B)) \\ &= m(\mathrm{Cl}(A)) + m(\mathrm{Cl}(B)) - m(\mathrm{Cl}(A) \cap \mathrm{Cl}(B)) \\ &= m(\mathrm{Cl}(A)) + m(\mathrm{Cl}(B)) = m(A) + m(B). \end{aligned}$$

It remains to notice that $\partial(A \cup B) \subset \partial A \cup \partial B$. Thus $A \cup B$ has null boundary and $m(\mathrm{Cl}(A \cup B)) = m(\mathrm{Int}\,\mathrm{Cl}(A \cup B))$, completing the proof. □

While restricting attention indeed resolves the issue of finite additivity of Lebesgue measure, it still doesn't resolved the issue of countable additivity. Clearly, each I_k as in the construction of $\mathcal{C}$ is in $\mathrm{RO}_0(\mathbb{R}^n)$, but as we already showed,

$$1/2 = \sum_{k \geqslant 1} m(I_k) \neq m(\textstyle\bigvee_{k \geqslant 1} I_k) = m(\mathrm{Int}\,\mathrm{Cl}(\bigcup_{k \geqslant 1} I_k)) = m([0,1]) = 1.$$

This means that the Boolean algebra of regular open sets with null boundaries form an incomplete Boolean algebra, and that its completion is once again $\mathrm{RO}(\mathbb{R}^n)$. Nevertheless, Lando and Scott have shown that this Boolean algebra has many nice properties. $\mathrm{RON}(\mathbb{R}^n)$ sits densely in $\mathrm{RO}(\mathbb{R}^n)$ in the algebraic sense. It satisfies Roeper's axiomatizations of *region-based topology*, a collection of axioms which Roeper proved to be necessary for any collection of pointless regions constructed via an equivalence relation defined on boundary points for pointy regions in a locally compact Hausdorff space. And like the previous section, Lando and Scott have also shown that the notion of points, along with the entire pointy topology of $\mathbb{R}^n$, can be identified via certain equivalence relations, and that $\mathrm{RON}(\mathbb{R}^n)$ is not isomorphic to $\mathrm{RON}(\mathbb{R}^m)$ for any $m \neq n$, essentially establishing the notion of dimensionality in a parameter-free approach.

Another more classical approach is proposed by Arntznius, where he considered Lebesgue measure algebra, the algebra of Borel subset of $\mathbb{R}^n$ modulo the ideal of Lebesgue null sets. In his own words, he blurs the differences in regions which "do not correspond to differences in actual physical space." If we define an equivalence relation $A \sim B$ if the symmetric difference $A \Delta B = (A \backslash B) \cup (B \backslash A)$ has measure zero, then naturally all regions in the same equivalence class have the same Lebesgue measure. Therefore, it is well-defined to let $\mu_0([A]) = m(A)$ where $[A]$ is the equivalence class containing A. Since we are dealing with Lebesgue measure algebra, we may relax our assumption on regular open sets and instead directly work on equivalence classes of Borel sets. To this end we redefine the notions of join, meet, and complement on Lebesgue measure algebra canonically:

$$[A] \wedge [B] \coloneqq [A \cap B] \qquad [A] \vee [B] \coloneqq [A \cup B] \qquad -[A] \coloneqq [A^c].$$

Since a countable union of null sets is still null, intuitively, μ_0 is a countably additive measure on the Lebesgue measure algebra. Indeed, given pairwise disjoint equivalence classes $\{[A_k]\}_{k \geqslant 1}$ (in the sense that $[A_i] \wedge [A_j] = [\varnothing]$), we define $B_1 = A_1$ and $B_k = A_k \backslash \bigcup_{i=1}^{k-1} A_i$ inductively. It follows that the B_k's are pairwise disjoint, and

$$A_k \Delta B_k = A_k \backslash B_k = A_k \cap \bigcup_{i=1}^{k-1} A_i = \bigcup_{i=1}^{k-1} A_k \cap A_i$$

which is a null set because $A_k \cap A_i$ has measure zero for all $i < k$. Therefore $[A_k] = [B_k]$ for each k, and

$$\begin{aligned}\mu_0(\bigvee_{k\geqslant 1}[A_k]) = \mu_0(\bigvee_{k\geqslant 1}[B_k]) = \mu_0([\bigcup_{k\geqslant 1} B_k]) && \text{(Completeness of } \vee\text{)}\\ = m(\bigcup_{k\geqslant 1} B_k) = \sum_{k\geqslant 1} m(B_k) = \sum_{k\geqslant 1}\mu_0([B_k]) && \text{(Since } \mu_0([A]) = m(A)\text{)}\\ = \sum_{k\geqslant 1}\mu_0([A_k]). && \text{(Since } [A_k] = [B_k]\text{)}\end{aligned}$$

This approach clearly solves the predicament of *Big Cantor* and *Small Cantor* as no closure/interior operator is induced during the process of $\vee$ and $\wedge$ with respect to Lebesgue measure algebra. In the measure theoretic sense, this is a satisfying result since we can recover all Borel measurable pointy functions up to a null set of difference.

Arntzenius's approach meets the conditions for Roeper's axiomatizations, but in order to evaluate its compatibility, a topology must first be given to the Lebesgue measure algebra. Arntzenius's construction turns out to satisfy only nine out of the ten axioms by Roeper, failing the last one, which roughly says "if A sits 'strictly inside' B, then there exists a C such that A sits 'strictly inside C, and C sits 'strictly inside' B." Note that this is essentially the condition for Topological Gunk.

Formally, for two Borel sets A, B, Arntzenius defined them to be **connected** if there exists a point p such that any open set containing p intersects both A and B in a non-null region. A region A sit "strictly inside" B if A is not connected with the complement of B.

Proof. To construct a counterexample that breaks Roeper's last axiom, let us once again consider the Fat Cantor Set $\mathcal{C}$. Our goal is to show that any non-null subset of $\mathcal{C}$ is connected to $\mathcal{C}^c$. Recall the definition of $d(x, S)$ at the beginning of the introduction of Fat Cantor Set. Let x be any point in any non-null subset of $\mathcal{C}$ and let $\epsilon > 0$ be given. It follows that there exists a sufficiently large k such that $d(x, \bigcup_{i=1}^k I_i) < \epsilon/2$. This means $(x - \epsilon/2, x + \epsilon/2) \cap \bigcup_{i=1}^k I_i$ is nonempty. Since $\bigcup_{i=1}^k I_i$ is a union of intervals, all with lengths $\geqslant 4^{-k} > 0$, it follows that $(x - \epsilon, x + \epsilon) \cap \bigcup_{i=1}^k I_i$ contains at least an interval of positive length. This proves that any arbitrary non-null subset of $\mathcal{C}$ is connected to $\mathcal{C}^c$, so $\mathcal{C}$ violates Roeper's last axiom, and therefore Arntzenius's model does not satisfy Topological Gunk. □