We now shift our attention back to the (complete) Boolean algebra of regular open sets on $\mathbb{R}^n$. In this section we attempt to recover abstract integration of real-valued random variables from a point-less perspective. This approach introduces a few complications: for example, without the mentioning of points, pointwise limits and convergence are not defined, so we need new modes of convergence. Like many standard treatments, we begin by defining indicator, simple, and elementary random variables, in that order. Then we show that it is possible to define a general random variable in the quotient space of elementary random variables modulo the ideal of certain zero-related random variables. Like how Arntzenius resolved the additivity of Lebesgue measure in the previous section, we show that this definition of general random variables is consistent as we move on to define expected values and other related properties.

The basic notions are analogous to that of a standard pointy probability theory. Let $\mathfrak{B}$ be Boolean algebra with $\mathbf{0} = \varnothing$ and $\mathbf{1}$, as well as join operator $\vee$, meet $\wedge$, and complement defined. We define a "gunky" **probability law** $\mathbb{P}$ on $\mathfrak{B}$ to be a real-valued function that is:

(1) Strictly positive: $\mathbb{P}(x) \geqslant 0$ and $\mathbb{P}(x) = 0$ if and only if $x = \mathbf{0}$, so that every region has positive "size";

(2) $\mathbb{P}(x) \leqslant 1$ and $\mathbb{P}(x) = 1$ if and only if $x = \mathbf{1}$; and

(3) $\mathbb{P}(x \vee y) = \mathbb{P}(x) + \mathbb{P}(y)$ if $x \wedge y = \varnothing$.

Having looked much into the structure of "gunky" regions, we cannot venture to impose assumptions that are two strong. Nevertheless, right away we can derive a list of intuitive results that align with the pointy probability:

- $\mathbb{P}$ is finitely additive.
- Define a partial order $\leqslant$ by non-strict set inclusion and $<$ its strict counterpart. Then if $x \leqslant y$ (resp. $x < y$), $\mathbb{P}(x) \leqslant \mathbb{P}(y)$ (resp. $\mathbb{P}(x) < \mathbb{P}(y)$).
- Boolean rings. We can define two more operations on $\mathfrak{B}$: $+$ and $\cdot$ so that $(\mathfrak{B}, +, \cdot)$ forms a ring. For $x, y \in \mathfrak{B}$, define $x + y$ to be the **symmetric difference** $(x \wedge -y) \vee (-x \wedge y)$, and define $x \cdot y$ to be $x \wedge y$. The zero of this ring corresponds to $\mathbf{0} = \varnothing$ and the multiplicative identity corresponds to $\mathbf{1}$. It is also worth noting that this ring is **idempotent**: $x \cdot x = x$ for all x.
- Difference: define $x - y$ to be the unique element z such that $z \wedge y = 0$ and $z \vee y = x$. It follows that if $x \leqslant y$ then $\mathbb{P}(x) + \mathbb{P}(y - x) = \mathbb{P}(y)$.
- Inclusion-exclusion. $\mathbb{P}(x) + \mathbb{P}(y) = \mathbb{P}(x \vee y) + \mathbb{P}(x \wedge y)$.

  Proof. Observe that $x \vee y = (x \wedge y) \vee (x - x \wedge y) \vee (y - x \wedge y)$, and that

  $$\mathbb{P}(x - x \wedge y) + \mathbb{P}(y - x \wedge y) = \mathbb{P}(x) + \mathbb{P}(y) - 2\mathbb{P}(x \wedge y).$$

  Rearranging the original equation gives the desired result. □

Of course, we would want stronger formulations of the gunky probability measure — right now it is neither countably additive nor guaranteed to be complete. To approach this, we will define a metric space $(\mathfrak{B}, d)$ from $(\mathfrak{B}, \mathbb{P})$ and define the completion $\overline{\mathfrak{B}}$ of $\mathfrak{B}$. Naturally, a distance metric on elements of $\mathfrak{B}$ measures how "far" elements are between each other, a natural candidate of which is the symmetric difference $+$. Hence we define $d(x, y) := \mathbb{P}(x + y)$.

It is easy to verify that $d(\cdot,\cdot)$ indeed defines a metric. $d(x,y)=0$ if and only if $\mathbb{P}(x+y)=0$, if and only if $x+y=\varnothing$, or equivalently $x=y$. Symmetry is clear. For triangle inequality, note that $a+b\leqslant a\vee b$, so

$$\begin{aligned} d(x,y)=\mathbb{P}(x+y)&=\mathbb{P}((x+y)+(z+z))=\mathbb{P}((x+z)+(y+z))\\ &\leqslant\mathbb{P}((x+z)\vee(y+z))=\mathbb{P}(x+z)+\mathbb{P}(y+z)=d(x,z)+d(y,z). \end{aligned}$$

With a metric defined, we obtain our first mode of convergence, which we call **$\mathfrak{B}$-convergence**. We say a sequence $\{x_n\}$ $\mathfrak{B}$-converges to a **$\mathfrak{B}$-limit** $x\in\mathfrak{B}$ if $d(x_n,x)\to 0$, or equivalently $\mathbb{P}(x_n+x)\to 0$. Similarly, we say $\{x_n\}$ is **$\mathfrak{B}$-Cauchy** if $\lim_{n\to\infty}\sup_{i,j\geqslant n}d(x_i,x_j)\to 0$, and like usual, we say $(\mathfrak{B},d)$ is **complete** if every $\mathfrak{B}$-Cauchy sequence also $\mathfrak{B}$-converges in the space. Let us reconsider the example of Fat Cantor Set in $\mathrm{RO}(\mathbb{R})$, where $\mathbb{P}$ is the Lebesgue measure. By construction, $d(\bigcup_{i=1}^{k-1}I_i,\bigcup_{i=1}^{k}I_i)=2^{-k}$, so $\{\bigcup_{i=1}^{k}I_i\}_{k\geqslant 1}$ forms a Cauchy sequence with respect to this metric space. Yet this sequence has no limit, since otherwise $\bigcup_{k\geqslant 1}I_k=I\backslash\mathcal{C}$ would have been an open regular set, which we have shown is false, since the interior of its closure is just $[0.1]$.

Our next goal, naturally, would be to complete $(\mathfrak{B},d)$, since it is well known that every metric space can be completed. The process has nothing exotic in it — it is a standard application of Cauchy completion.

We consider $\mathfrak{C}$, the collection of $\mathfrak{B}$-Cauchy sequences in $(\mathfrak{B},d)$. Let us first clarify the algebraic structures defined on $\mathfrak{C}$. Two sequences $\{x_n\},\{y_n\}$ are considered the same if they agree term-wise. Viewing $\mathfrak{C}$ as a Boolean ring, the operations are defined via:

- Multiplicative identity (one): the constant sequence of **1**'s.
- Additive identity (zero): the constant sequence of $\mathbf{0}=\varnothing$'s.
- Addition is defined term-wise: $\{x_n\}+\{y_n\}=\{x_n+y_n\}$ (where the latter is addition defined on $(\mathfrak{B},+,\cdot)$).
- Multiplication is defined term-wise too: $\{x_n\}\cdot\{y_n\}=\{x_n\cdot y_n\}$ ($\cdot$ from $(\mathfrak{B},+,\cdot)$ too).

It follows that if we define $\vee$ and $\wedge$ on $\mathfrak{C}$ by

$$\{x_n\}\wedge\{y_n\}:=\{x_n\}\cdot\{y_n\}\qquad\text{and}\qquad\{x_n\}\vee\{y_n\}:=\{x_n\}+\{y_n\}+\{x_n\}\cdot\{y_n\},$$

then

$$\{x_n\}\vee\{y_n\}=\{x_n\vee y_n\}\qquad\{x_n\}\wedge\{y_n\}=\{x_n\wedge y_n\}\qquad\{x_n\}^c=\{\mathbf{1}\}+\{x_n\}=\{x_n^c\}.$$

That is, $\mathfrak{C}$ can be viewed as a Boolean algebra with respect to these operations.
We say two Cauchy sequences $\{x_n\},\{y_n\}$ are **co-Cauchy** if $d(x_n,y_n)\to 0$. Define $\overline{\mathfrak{B}}$ to be $\mathfrak{C}$ modulo the equivalence relation of being co-Cauchy. Alternatively, this can be characterized by $\mathfrak{C}$ modulo the ideal $\mathfrak{C}_0$ of sequences with $\mathfrak{B}$-limit $\varnothing$. We write $\overline{\mathfrak{B}}=\mathfrak{C}/\mathfrak{C}_0$. Finally, we define a new metric $\overline{d}$ on $\overline{\mathfrak{B}}$ by

$$\overline{d}(x,y)=\overline{d}([\{x_n\}],[\{y_n\}]):=\lim_{n\to\infty}d(x_n,y_n).$$

This is a well-defined metric because of how we constructed our equivalence classes: if $\{y_n\}$ and $\{z_n\}$ are co-Cauchy and $\lim d(x_n,y_n)\to 0$, then $\lim(x_n,z_n)\leqslant\lim d(x_n,y_n)+\lim d(y_n,z_n)=\lim d(x_n,y_n)=0$. It is, therefore, natural to define a probability law $\overline{\mathbb{P}}$ on $\overline{\mathfrak{B}}$ by

$$\overline{\mathbb{P}}([\{x_n\}]):=\lim_{n\to\infty}\mathbb{P}(x_n).$$

Theorem

$\overline{\mathfrak{B}}$ is complete with respect to $\overline{d}$, and $\overline{\mathbb{P}}$ is a countably additive probability measure on $\overline{\mathfrak{B}}$.

Proof that $\overline{\mathfrak{B}}$ is complete. Let $\{[C_k]\}_{k\geqslant 1}$ be a Cauchy sequence on $(\overline{\mathfrak{B}},\overline{d})$. Note by definition each C_k is a *sequence* of $\mathfrak{B}$. For each k, discard early terms of C_k, leaving C'_k, the collection of sufficiently late terms so that

$$\sup_{x,y\in C'_k} d(x,y) < \frac{1}{k}.$$

Since a subsequence of a Cauchy sequence is co-Cauchy with the original sequence, for each k, C_k and C'_k belong to the same equivalence class, and therefore $\{[C_k]\}_{k\geqslant 1} = \{[C'_k]\}_{k\geqslant 1}$. Define $c_{k,n}$ to be the n^{th} term of the modified sequence C'_k. Now construct a sequence $X = \{x_n\} \subset \mathfrak{B}$ by setting $x_k = c_{k,k}$, the diagonal sequence of $\{C'_k\}$. The proof is done if we show that $X \in \mathfrak{C}$ and that $\{[C_k]\}$ converges to $[X]$ with respect to $(\overline{\mathfrak{B}},\overline{d})$.
To show X is $\mathfrak{B}$-Cauchy, let $\epsilon > 0$ be given and let N be sufficiently large so that $\sup_{k,j>N} \overline{d}(C_k,C_j) < \epsilon/3$. Then, for $k,j > N$ and any n,

$$\begin{aligned} d(x_k,x_j) = d(c_{k,k},c_{j,j}) &\leqslant d(c_{k,k},c_{k,n}) + d(c_{k,n},c_{j,n}) + d(c_{j,n},c_{j,j}) \\ &\leqslant 1/k + 1/j + d(c_{k,n},c_{j,n}). \end{aligned}$$

Assuming $N > 3\epsilon^{-1}$, $1/k + 1/j < 2\epsilon/3$. On the other hand by assumption $\overline{d}(C_k,C_j) = \lim_n d(c_{k,n},c_{j,n}) < \epsilon/3$, so for sufficiently large n, $d(c_{k,n},c_{j,n}) < \epsilon/3$ as well. Since the above triangle inequality holds for arbitrary n, we conclude that $d(x_k,x_j) < \epsilon$, and therefore X is $\mathfrak{B}$-Cauchy.
Finally, to show $\{[C_k]\}$ converges to $[X]$, for any $\epsilon > 0$, we pick N sufficiently large so that $\sup_{k,j>N} d(x_k,x_j) < \epsilon/2$. Then

$$d(c_{k,j},x_j) \leqslant d(c_{k,j},c_{k,k}) + d(c_{k,k},x_j) = d(c_{k,j},c_{k,k}) + d(x_k,x_j) \leqslant 1/k + \epsilon/2 < \epsilon$$

if $N > 2\epsilon^{-1}$. Taking limit in j, then in k, we see $\overline{d}([C_k],[X]) = 0$, completing the proof that $\overline{\mathfrak{B}}$ is complete. □

Proof that $\overline{\mathbb{P}}$ is a countably additive probability measure. It is clear that $\overline{\mathbb{P}}$ is a probability measure, since it inherits the structure of $\mathbb{P}$. So we will focus on proving its countable additivity.
Let us first show that $\overline{\mathbb{P}}$ is finitely additive. In particular let $[\{x_n\}]$ and $[\{y_n\}]$ be given. Assume they are disjoint in the sense that $[\{x_n\}] \wedge [\{y_n\}] = [\{\varnothing\}]$. Proving additivity in this case is straightforward by definition:

$$\begin{aligned} \overline{\mathbb{P}}([\{x_n\}] \vee [\{y_n\}]) = \overline{\mathbb{P}}([\{x_n \vee y_n\}]) &= \lim_{n\to\infty} \mathbb{P}(x_n \vee y_n) && \text{(by definition)} \\ &= \lim_{n\to\infty} (\mathbb{P}(x_n) + \mathbb{P}(y_n) - \mathbb{P}(x_n \wedge y_n)) && \text{(Inclusion-exclusion)} \\ &= \lim_{n\to\infty} \mathbb{P}(x_n) + \lim_{n\to\infty} \mathbb{P}(y_n) - \underbrace{\lim_{n\to\infty} \mathbb{P}(x_n \wedge y_n)}_{=0} = \overline{\mathbb{P}}([\{x_n\}]) + \overline{\mathbb{P}}([\{y_n\}]) \end{aligned}$$

To prove countable additivity, let $\{[C_k]\} \subset \overline{\mathfrak{B}}$ be pairwise disjoint, where each C_k is a $\mathfrak{B}$-Cauchy sequence. We assume that their infinite join $[C] = \bigvee_{k\geqslant 1}[C_k]$ exists in $\overline{\mathfrak{B}}$, and the goal is to show $\overline{\mathbb{P}}([C]) = \sum_{k\geqslant 1} \overline{\mathbb{P}}([C_k])$. It is known in standard pointy measure theory that finite additivity, combined with bounded continuity from above, implies countable additivity. Here we adopt a similar approach. To this end, let us consider another sequence with $[X_k] \geqslant [X_{k+1}]$ for each k, and that $\bigwedge_{k\geqslant 1}[X_k] = [\{\varnothing\}]$ (with respect to $\overline{d}$).

Monotonicity of $\overline{\mathbb{P}}$ implies that $\lim \overline{\mathbb{P}}([X_k])$ exists and the sequence is in particular Cauchy. Since $\overline{\mathfrak{B}}$ is complete, $[X_k]$ also $\overline{\mathfrak{B}}$-converges to some limit which we call $[\overline{X}]$. Our goal is, of course, to show that $[\overline{X}] = [\{\varnothing\}]$, but this is obvious: for each $k \geqslant 1$,

$$\begin{aligned}
[\overline{X}] \wedge [X_k] &= \lim_{n\to\infty} [X_n] \wedge \lim_{n\to\infty} [X_k] && \text{(treating } \{[X_k]_{n\geqslant 1}\} \text{ as constant sequence in } n\text{)} \\
&= \lim_{n\to\infty} ([X_n] \wedge [X_k]) \\
&= \lim_{n\to\infty} [X_n] = [\overline{X}]. && \text{(} [X_k] \text{ is monotonically decreasing)}
\end{aligned}$$

This means $[\overline{X}] \leqslant [X_k]$ for each k. Since $\bigwedge_{k\geqslant 1}[X_k] = [\{\varnothing\}]$ we conclude that $[\overline{X}] = [\{\varnothing\}]$, and so $\overline{\mathbb{P}}([X_k]) = 0$. This shows continuity of $\overline{\mathbb{P}}$ at the zero of $\overline{\mathfrak{B}}$.

To complete the proof, intuitively we can "translate" and "invert" the monotonicity and limit. Formally, we can define $[X_k] \coloneqq [C] + \bigvee_{i\leqslant k}[C_i]$ and $[X] = \bigwedge_{k\geqslant 1}[X_k]$. Here, X_k is the "tail join" of $[C_k]$'s that is disjoint from the early part $\bigvee_{i\leqslant k}[C_i]$, and $[X]$ in some informal sense the $\liminf$ of the $[C_k]$'s. If $[X] \neq [\{\varnothing\}]$ then it is disjoint from any of the $[C_i]$'s, so $[X] \wedge [C] = [\{\varnothing\}]$. Since $[X_k] \leqslant [C]$ it follows that $[X] \wedge [X_k] = [\{\varnothing\}]$ as well. But on the other hand $[X]$ is defined to be the infinite meet of the $[X_k]$'s, so $[X] \wedge [X_k] = [X]$ for every k. Therefore $[X] = [\{\varnothing\}]$, and by continuity at zero, $\overline{\mathbb{P}}([X]) = 0$. Using definition,

$$0 = \overline{\mathbb{P}}([X]) = \lim_{k\to\infty} \overline{\mathbb{P}}([X_k]) = \lim_{k\to\infty} \overline{\mathbb{P}}([C] + \bigvee_{i=1}^{k}[C_i]) = \overline{\mathbb{P}}([C]) - \lim_{k\to\infty} \overline{\mathbb{P}}(\bigvee_{i=1}^{k}[C_i])$$

and the proof is complete! $\square$