Region-Based Topology

Author(s): Peter Roeper

Source: *Journal of Philosophical Logic*, Jun., 1997, Vol. 26, No. 3 (Jun., 1997), pp. 251-309

Published by: Springer

Stable URL: https://www.jstor.org/stable/30227095

PETER ROEPER

REGION-BASED TOPOLOGY[1]

ABSTRACT. A topological description of space is given, based on the relation of *connection* among regions and the property of being *limited*. A minimal set of 10 constraints is shown to permit definitions of points and of open and closed sets of points and to be characteristic of locally compact T_2 spaces. The effect of adding further constraints is investigated, especially those that characterise continua. Finally, the properties of mappings in region-based topology are studied. Not all such mappings correspond to point functions and those that do correspond to continuous functions.

It is taken for granted that points are not parts or elements of space; a point is a location in space. As a consequence, points are not the primary bearers of spatial properties and spatial relations, nor the primary objects of spatial mappings. This role belongs rather to the parts of space. As is common practice, I shall use the term 'region' as an expression applicable to the parts of any kind of space, irrespective of the number of dimensions. And I shall diverge somewhat from ordinary usage by acknowledging as regions parts of space that are "not in one piece", that consist e.g. of areas that are disconnected from one another. Since regions are the primary bearers of spatial properties and relations, it should be possible to describe spatial structures in terms of relationships among regions, and it should be possible to identify the points of a space by means of its structure, and hence in terms of the regions in which they are located. It is the purpose of this paper to pursue these two aims in the case of topological structure. Obviously, the concept of *topology* used here is an intuitive one and appeal is made to an understanding of topological structure that is independent of the mathematical theory of (point-based) topology.

In Section 1 two primitive notions are introduced which are required for the topological description of spaces in terms of regions. They are the relation of *connection* among regions and the property of being *limited*. Then ten constraints are formulated and adopted as characteristic of *region-based topologies*. Sections 2 and 3 deal with the specification of points in terms of regions. Points are identifiable by means of certain families of filters whose elements are regions or, alternatively, by certain classes of regions. The different characterisations are shown to be equivalent.

Journal of Philosophical Logic **26:** 251–309, 1997.

Having specified points within the conceptual framework of region-based topology the next step consists in characterising open and closed sets of points in the same way. This is done Section 4. It is also shown there that the points in a region-based topology constitute a locally compact T_2 space and that regions correspond to regular-closed sets of points. The aim of Section 5 is to generate from a topological structure that is described in terms of open and closed sets of points a region-based topology which is isomorphic to the space (described in terms of regions and their relations) in which those points are locations. The section concludes with a discussion of the results obtained so far.

Sections 6 and 7 study more specific kinds of region-based topologies by considering further constraints that might be added to those formulated in Section 1. Section 6 concentrates on the notion of *continuity* (of a space), while Section 7 is concerned with the characteristics of a *continuum* and with the definability of the second primitive notion in region-based topology, *limitedness*, in terms of the first, *connection*.

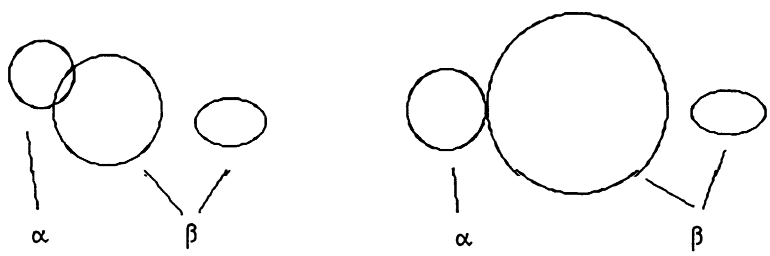
The final section, Section 8, deals with mappings from one region-based topology into another and ends with a discussion of the results, the most important of which is that if such a mapping corresponds to a function from the points in the first topology to the points in the second, then that function is continuous.

1. Characterisation of Region-Based Topologies

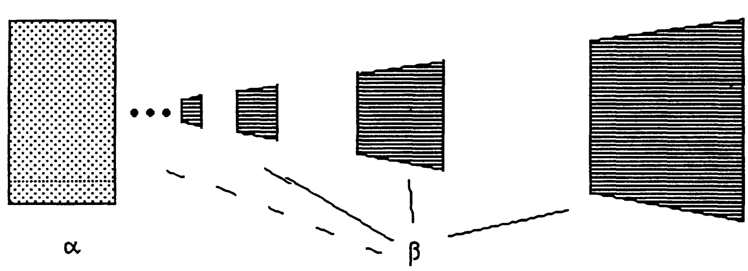
The space of a topology is represented by a non-degenerate Boolean algebra Ω,[2] whose elements are the regions making up the space, so that the maximal element 1 is the space itself. This is prompted by the thought that the regions as parts of the space are subject to the usual mereological relations and operations: '$\alpha \leq \beta$' signifies that α is a part or subregion of β and '$\alpha < \beta$' that α is a proper subregion of β: the *join* $\alpha \vee \beta$ of two regions α and β is the smallest region having both α and β as parts; the *meet* $\alpha \wedge \beta$ is the largest region common to α and β, and the *complement* $-\alpha$ of a region α is the region that comprises all of the space except α. It should be clear from these explications that a region, as understood here, need not be coherent; a region may consist of separated subregions, even infinitely many of them. So, the regions of a space constitute a mereological structure, a structure appropriately described by a set of constraints known as *mereology*. However, mereology does not recognise a null-element, and as a consequence the meet and complement operations are not always defined. Now our normal understanding of 'region' does indeed not allow for a null-region. So, it is merely for the purpose of simplifying formal arguments that I add the null-element to the mereological structure, thereby turning it into a Boolean algebra.[3]

In addition to the operations mentioned so far I will also make use of the *infinite join* $\vee\Sigma$ of a set Σ of regions, the smallest region having every region in Σ as a part, and the *infinite meet* $\wedge\Sigma$, the largest region common to all the regions in Σ. In standard Boolean algebras $\vee\Sigma$ and $\wedge\Sigma$ are not guaranteed to exist in every case and indeed spaces appear conceivable in which an infinity of regions does not always make up one single region and does not always have a largest common subregion. So, when $\vee\Sigma$ and $\wedge\Sigma$ *are* assumed to exist for every Σ, in which case the Boolean algebra is called *complete*, this will be made explicit.[4] In the later parts of this paper the Boolean algebra Ω, which underlies the topological structure will be assumed to be complete; but in the first five sections the completeness of Ω is assumed only when assumed explicitly.

The topological structure of the space is embodied in features that are superimposed on the Boolean structure. The basic concept to be employed, the first of two notions that are taken as primitive, is the 2-place relation of *connection*, to be symbolised by '∞'. The paradigmatic cases of connection are those where two regions either overlap or are in contact with one another, in as much as a coherent subregion of one either overlaps with or is in contact with a coherent subregion of the other. This is the case with α and β in the following two examples.



But '∞' is also intended to cover cases like this:



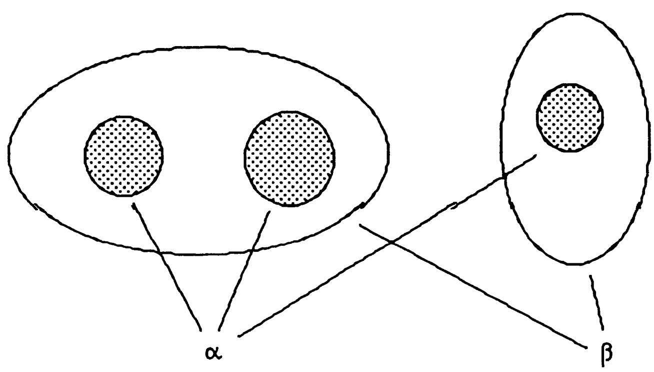
Here no coherent subregion of β is in contact with α. However, β is not separated from α by any (positive) distance. (Although *distance* itself is not a topological notion, *having a positive distance* is.) The two regions are *infinitesimally close* and there is a point which is a boundary point of α as well as β. So, α will count as *connected with* β when α and β overlap, or are in contact, or are at least infinitesimally close to one another.

When α and β are connected, then there is at least on point that is located on or, as I shall say, coincident with both α and β. The system of constraints A1 to A10 to be introduced in this section will permit derivation of this result. However, the system of constraints does allow interpretations in which $\alpha \infty \beta$ does not hold in all cases in which α and β are infinitesimally close, while not in contact. A further constraint B4, which is to be added in Section 6, will exclude such interpretations and guarantee that α is connected with β whenever α and β are infinitesimally close.

In terms of *connection* three further notions with intuitive significance are definable, which all have a role to play in region-based topology.

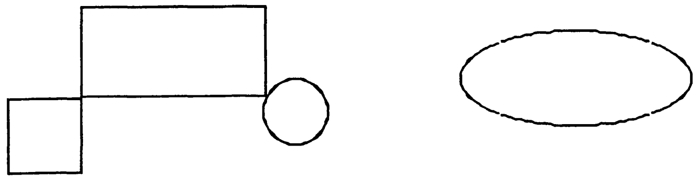
DEFINITION 1.1. $\alpha \ll \beta$ if and only if $\alpha \not\infty -\beta$.

$\alpha \ll \beta$ typically holds when α is an interior part of β, when α is completely included in β, in such a way that no point is a boundary point of both α and β, as for example in the following diagram.



When $\alpha \neq 0$, $\alpha \ll \alpha$ will hold only when α is disconnected from the rest of the space, i.e. when α is a *component* of the space or consists of components.

The other two notions capture different ways in which a region may be *in one piece*, or (*self-*)*connected*. I shall use the terms *coherence* and *convexity* for the situations illustrated by α and β, respectively.



α β

In a *coherent* region one can get from anywhere to anywhere else without going outside the region, hence a coherent region does not consist of two or more disconnected regions. Both α and β are coherent; but the combined region $\alpha \vee \beta$ is not, as it consists of the regions α and β, which are not connected with one another. A *convex* region allows one to go from anywhere within the region to anywhere else within the region without crossing a boundary. So, β is convex, but α is not.

DEFINITION 1.2. α is *coherent* for: if β and γ are any non-null regions such that $\beta \vee \gamma = \alpha$ then $\beta \infty \gamma$.

DEFINITION 1.3. α is *convex* for: if β and γ are non-null regions such that $\beta \vee \gamma = \alpha$, then there is a region α' such that $\alpha' \ll \alpha$ and $(\alpha' \wedge \beta) \infty (\alpha' \wedge \gamma)$.

It is obvious from these definitions that when a region is convex it is coherent. Not every coherent region is also convex. But if the region in question is the whole space the two notions coincide. In 1-dimensional spaces, too, the two notions coincide.

The list of axioms begins with five that capture central characteristics of *connection*.

A1 *If* $\alpha \infty \beta$ *then* $\beta \infty \alpha$.
A2 *If* $\alpha \neq \beta$ *then* $\alpha \infty \alpha$.
A3 $0 \not\infty \alpha$.
A4 *If* $\alpha \infty \beta$ *and* $\beta \leq \gamma$ *then* $\alpha \infty \gamma$.
A5 *If* $\alpha \infty (\beta \vee \gamma)$ *then* $\alpha \infty \beta$ *or* $\alpha \infty \gamma$.

The second primitive notion employed is that of being a *limited* region, i.e. a region that is limited – in all directions – by a boundary. A limited 1-dimensional region, e.g. a finite interval, is limited by points; a limited 2-dimensional region, such as a circle, is limited by a line or lines. Of course, the notion of limitedness is applicable not only to the space as a whole but also to the regions constituting the space; limited regions have boundaries in all directions. And wherever a point is located there is the boundary of some region or other.

The regions in a limited 1-dimensional space (a finite line), with '∞' understood as intended (i.e. $\alpha \infty \beta$ when α and β are at least infinitesimally close) meet constraints A1–A5; and so do the regions in an unlimited 1-dimensional space (an infinite line). So, the constraints adopted so far leave it open whether the space is limited or unlimited. Furthermore, there is no obvious way of defining limitedness in terms of connection without adding to the constraints governing connection. On the other hand, since points are located on the boundaries of regions, limitedness plays an essential role in the characterisation of points in Section 2. So, seeing that it is a topological concept that needs to be taken into account, it is treated here as a primitive. The question of definability will be taken up again in Section 7. The basic characteristics of limitedness, exhibiting formal similarities with A3–A5, are these:

A6 0 *is limited.*
A7 *If* α *is limited and* $\beta \leq \alpha$ *then* β *is limited.*
A8 *If* α *and* β *are both limited then* $\alpha \vee \beta$ *is limited.*

Two further constraints involve both notions, connection and limitedness. The intended content of the first is that a region that is connected with another region is limited where it is so connected. And the though behind this is that when two regions are connected then there is at least one point coincident with both, and where a point is located a region must be limited.

A9 *If* $\alpha \infty \beta$ *then there is a limited region* β' *with* $\beta' \leq \beta$ *and* $\alpha \infty \beta'$.

Finally, there is a separability or interpolation constraint.

A10 *If* α *is limited,* $\beta \neq 0$, *and* $\alpha \ll \beta$, *then there is a region* γ *such that* $\gamma \neq 0$, γ *is limited, and* $\alpha \ll \gamma \ll \beta$.

Constraint A10 postulates a certain kind of divisibility of the space, akin to but not the same as infinite divisibility in the sense that for every region there is a proper subregion. Infinite divisibility will be addressed in Section 6. A10 plays a crucial role in forming equivalence classes of ultrafilters which can represent points. And it turns out that in the absence of constraint A10 a 2-place relation of connection may not be sufficient to describe topological structure.[5]

Postulates A1–A10 form the core of region-based topology. In the next sections structures satisfying these constraints will be studied. In order to have a convenient expression I shall talk of *region-based topologies*, meaning by a region-based topology $\mathcal{R}$ a structure $\langle \Omega, \infty, \Delta \rangle$, where Ω is a non-degenerate Boolean algebra of regions, ∞ the 2-place relation of connection and Δ the set of limited regions, the latter two being subject to constraints A1 to A10.

The remainder of this section lists some consequences of the constraints which will be appealed to in later parts of the paper.

LEMMA 1.1. *If* $\alpha \wedge \beta \neq 0$ *then* $\alpha \infty \beta$.

Proof. Suppose $\alpha \wedge \beta \neq 0$. Then $(\alpha \wedge \beta) \infty (\alpha \wedge \beta)$ by constraint A2. Hence $\alpha \infty \beta$ by A4 and A1. □

LEMMA 1.2. *If* $\alpha \not\infty \beta$ *then* $\alpha \leq -\beta$. *(If* $\alpha \ll \beta$ *then* $\alpha \leq \beta$.*)*

Proof. Assume not $\alpha \leq -\beta$. Then $(\alpha \wedge \beta) \neq 0$ and so $\alpha \infty \beta$ by Lemma 1.1. □

LEMMA 1.3. (a) *If* $\alpha \neq 0$ *then there is a limited region* $\beta \neq 0$ *with* $\beta \ll \alpha$;

(b) *If* $\alpha \neq 0$ *then there is a limited region* $\beta \neq 0$ *with* $\beta \leq \alpha$;

(c) $\alpha = \mathrm{V}\{\beta \mid \beta$ is limited and $\beta \leq \alpha\}$.

Proof. (a) Assume $\alpha \neq 0$. Since $0 \ll \alpha$ by A3 and 0 is limited by A6, there is, by A10, a limited region $\beta \neq 0$ with $\beta \ll \alpha$. (b) By (a) and Lemma 1.2. (c) By (b) and A6. □

LEMMA 1.4. *If* α *is limited and* $\alpha \not\infty \beta$ *then there is a region* γ *such that* $\alpha \not\infty -\gamma$ *and* $\gamma \not\infty \beta$. *(If* α *is limited and* $\alpha \not\infty \beta$ *then there is a region* γ *such that* $\alpha \ll \gamma$ *and* $\beta \ll -\gamma$.*)*

Proof. Assume that α is limited and $\alpha \not\infty \beta$, i.e. $\alpha \ll -\beta$. Then by A10 there exists a region γ with $\alpha \ll \gamma \ll -\beta$; i.e. $\alpha \not\infty -\gamma$ and $\gamma \not\infty \beta$. □

LEMMA 1.5. (a) *If* α *is limited then there is a limited region* γ *such that* $\alpha \ll \gamma$;

(b) *If* α *is limited,* β *unlimited and* $\alpha \leq \beta$, *then there is a limited region* γ *such that* $\alpha < \gamma < \beta$;

(c) *If* α *is limited and* 1 *unlimited, then there is a limited region* γ *such that both* $\alpha \ll \gamma$ *and* $\alpha < \gamma$.

Proof. (a) Suppose α is limited. $\alpha \ll 1$ by A3. So, by A10 there is a limited region γ with $\alpha \ll \gamma$. (b) Suppose α limited, β unlimited, and $\alpha \leq \beta$. Then $\beta \wedge -\alpha \neq 0$ is unlimited by A8. By Lemma 1.3(b) there is a limited region $\alpha' \neq 0$ with $\alpha' \leq (\beta \wedge -\alpha)$ and hence $\alpha' < (\beta \wedge -\alpha)$. So, $\gamma = \alpha \vee \alpha'$ is limited by A8 and $\alpha < \gamma < \beta$. (c) by (a), (b) and A8. □

LEMMA 1.6. *If* α *is limited and* $\alpha \infty \beta$ *then there is a limited region* $\beta' \leq \beta$ *such that* $\alpha \infty \beta'$ *and* $\alpha \not\infty (\beta \wedge -\beta')$.

Proof. Suppose α limited and $\alpha \infty \beta$. Then by Lemma 1.5(a) there is a limited γ with $\alpha \not\infty -\gamma$. Let β' be $\beta \wedge \gamma$. Then β' is limited by A7 and $\alpha \infty \beta'$ by A5. And since $\beta \wedge -\beta' = \beta \wedge -\gamma$, $\alpha \not\infty (\beta \wedge -\beta')$ by A4. □

LEMMA 1.7. 0 *is convex.*

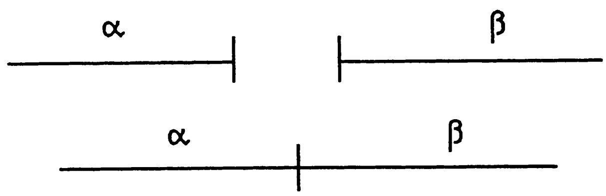
LEMMA 1.8. 1 *is coherent if and only if* $\alpha \infty -\alpha$ *for every region* α *with* $\alpha \neq 0$ *and* $\alpha \neq 1$.

Proof by Lemma 1.1. □

2. Points

The guiding thought of the present approach is that the topology of a space is fully determined by the relationships of connection holding among the regions that are parts of the space. Part of the thought is that these relationships also determine which points there are in the space. And this implies that points are characterisable in terms of region. There are several proposals concerning such characterisation including those by Whitehead, by Menger and by Tarski.[6] In almost all cases a point is characterised by way of certain of the regions that it is located in, where the set of those regions conforms to some limit process.[7] The variant of this approach adopted here employs certain sets of ultrafilters[8] on the Boolean algebra of regions, the ultrafilters consisting of regions with which the point in question is coincident.

Of course, the notion of an ultrafilter, i.e. a *maximal* filter, brings with it the idea of a limit process; but ultrafilters themselves cannot, in general, do duty for points.[9] Membership conditions for ultrafilters are formulated exclusively in terms of Boolean relations and operations, so that the ultrafilters are independent of the topology imposed on the Boolean algebra. The identity of points, on the other hand, is clearly dependent on the topology of the space whose parts constitute the Boolean algebra. A simple example may serve as illustration:



Whether the boundaries of line α and line β constitute one point or two depends on the relationship of connection holding or not holding between α and β. So it is not individual ultrafilters but sets of collocated ultrafilters that serve as points, and which ultrafilters are collocated is a function of the topology.

Sets of ultrafilters are not the intuitively most plausible means of representing points. One would rather be inclined to think of the totality of regions that a point is coincident with as a suitable entity to play the role of the point. My choice of certain sets of ultrafilters as substitutes for points is prompted in part by formal considerations; it allows e.g. a simple characterisation of open and closed sets of points in Section 4. But there is also the theoretical aspect that it makes transparent how the identity of points depends on the combination of mereological and topological features of the space: ultrafilters, being constituted by virtue of the *mereological* relations among regions serve as *mereological limits*, while the *collocation* of ultrafilters is entirely due to the *topological* relations among regions. The representation of a point directly by means of the regions it is coincident with is not ignored; it will be considered in Section 3. This is not a metaphysical issue; I am not proposing a *reduction* of points to certain sets of sets of regions. I cannot offer anything more concerning the nature of points than what has already been said: they are locations in space, not constituents or parts of space. The sets that have been mentioned should be thought of as means by which points can be identified. In short, a point is identified *by* such a set, not *with* it. Having said this, I hope I will be excused if in what follows I use less careful language, and talk of sets of ultrafilters as if they *were* points. The characterisation of points in terms of sets of ultrafilters will be used in Section 4 to arrive at a structure of point-based topology.

The first task in the process leading to the definition of *point* is to extend the relation of connection to filters in this evident way:

DEFINITION 2.1. Let Φ be a filter on the Boolean algebra Ω. Then Φ is said to be *connected with* region α ($\Phi \infty \alpha$) if, for every element β of Φ, $\beta \infty \alpha$.

DEFINITION 2.2. Let Φ_1 and Φ_2 be filters on the Boolean algebra Ω. Then Φ_1 is said to be *connected with* Φ_2 ($\Phi_1 \infty \Phi_2$) if, for all elements α of Φ_1 and β of Φ_2, $\alpha \infty \beta$. If Φ_1 and Φ_2 are ultrafilters I also say in this case that they are *collocated.*

In a similar way the notion of *limitedness* is extended so as to be applicable to filters.

DEFINITION 2.3. A filter Φ is said to be *limited* if at least one region α in Φ is limited.

Whether two ultrafilters are connected depends evidently on the topology added to Ω. It conforms to intuition to regard several ultrafilters as located together if they are mutually connected. And so, under different topologies, different ultrafilters will be located together. If is natural then to take as substitutes for points maximal classes of mutually connected ultrafilters.

As I have said before, the elements of the ultrafilters in such a class E are regions that the point to be identified is coincident with. But any point is coincident with at least one *limited* region, as the justification for constraint A9 implies. So, at least one of the ultrafilters belonging to E must be limited. Now Theorem 2.1 below establishes that if one ultrafilter in E is limited then all are. And so points are to be represented by maximal classes of *limited* ultrafilters that are mutually connected. Theorem 2.2 establishes that the relation of connection is an equivalence relation for limited ultrafilters. And so we finally arrive at the representation of points by *equivalence classes under connection of limited ultrafilters*.

LEMMA 2.1. *Let α be a region and ∇ an ultrafilter on Ω. Then if α is limited and $\alpha \infty \nabla$, ∇ is limited.*

Proof. Suppose α is limited, and $\alpha \infty \nabla$. Then by Lemma 1.6 there exists for any $\beta \in \nabla$ a limited $\beta' \leq \beta$ such that $\alpha \not\infty (\beta \wedge -\beta')$. Since ∇ is an ultrafilter, either $\beta' \in \nabla$ or $(\beta \wedge -\beta') \in \nabla$. So, given that $\alpha \infty \nabla$, $\beta' \in \nabla$. Hence ∇ is limited. □

THEOREM 2.1. *If ∇ is limited and $\nabla \infty \nabla'$, then ∇' is limited.*

Proof. Suppose ∇ is limited and $\nabla \infty \nabla'$. Then there exists a limited region $\alpha \in \nabla$. $\alpha \infty \nabla'$ by Definition 2.2. Hence ∇' is limited by Lemma 2.1. □

LEMMA 2.2. *For every ultrafilter ∇, $\nabla \infty \nabla$.*

Proof. Let ∇ be an ultrafilter. Then $\alpha \wedge \beta \neq 0$ for any elements α and β of ∇, since ∇ is a proper filter. Hence $\alpha \infty \beta$ by Lemma 1.1. So $\nabla \infty \nabla$. □

LEMMA 2.3. *For ultrafilters ∇_1 and ∇_2, $\nabla_1 \infty \nabla_2$ if $\nabla_2 \infty \nabla_1$.*

Proof by constraint A1. □

LEMMA 2.4. *If α is limited, $\alpha \infty \nabla$, and $\nabla \infty \beta$, then $\alpha \infty \beta$.*

Proof. Suppose that α is limited, $\alpha \infty \nabla$, and $\nabla \infty \beta$, but that $\alpha \not\infty \beta$. Then by Lemma 1.4 there exists a region γ with $\alpha \not\infty -\gamma$ and $\gamma \not\infty \beta$. Since ∇ is an ultrafilter, either $\gamma \in \nabla$ or $-\gamma \in \nabla$. But then either $\alpha \infty -\gamma$ or $\gamma \infty \beta$, q.e.a. Hence $\alpha \infty \beta$. □

LEMMA 2.5. *Let* ∇ *and* ∇' *be ultrafilters,* ∇ *limited. Then* $\nabla \infty \nabla'$ *if* $\gamma \infty \beta$ *for every limited* $\gamma \in \nabla$ *and every* $\beta \in \nabla'$.

Proof. Assume that γ is limited, that α and γ are in ∇, and that β is in ∇'. Then $\alpha \wedge \gamma$ is in ∇. $\alpha \wedge \gamma$ is limited by constraint A7, and $(\alpha \wedge \gamma) \infty \beta$ by assumption. So $\alpha \infty \beta$ by A1 and A4. Hence $\nabla \infty \nabla'$. □

LEMMA 2.6. *Let* ∇_1, ∇_2, *and* ∇_3 *be limited ultrafilters. Then* $\nabla_1 \infty \nabla_3$ *if* $\nabla_1 \infty \nabla_2$ *and* $\nabla_2 \infty \nabla_3$.

Proof. Suppose that $\nabla_1 \infty \nabla_2$ and $\nabla_2 \infty \nabla_3$. Then $\alpha \infty \beta$ for every limited element α of ∇_1 and every element β of ∇_3 by Lemma 2.4. So $\nabla_1 \infty \nabla_3$ by Lemma 2.5. □

THEOREM 2.2. *The relation of connection is an equivalence relation among limited ultrafilters.*

Proof by Theorem 2.1 and Lemmas 2.2, 2.3, and 2.6. □

So, as anticipated, I adopt this formal definition of *point*:

DEFINITION 2.4. Let ∇ be a limited ultrafilter on Ω. Then $\{\nabla' \mid \nabla' \infty \nabla\}$ (or $[\nabla]_\infty$), the equivalence class of ∇ under connection, is a *point*.

If $\mathcal{R} = \langle \Omega, \infty, \Delta \rangle$ is a region-based topology, then, by Lemma 1.3, there are limited non-null regions in Ω. So there are limited ultrafilters on Ω, and consequently there are points associated with $\mathcal{R}$. So, given a region-based topology $\mathcal{R}$, one can identify the points located in the space of the topology as the set $P_\mathcal{R}$ of equivalence classes under connection of limited ultrafilters, so that, for any limited ultrafilter ∇, $[\nabla]_\infty$ belongs to $P_\mathcal{R}$ and identifies a point. This definition of *point* also leads directly to a specification of the coincidence relation between points and regions. Suppose α is a region which is a member of at least one ultrafilter in $[\nabla]_\infty$, i.e. $\alpha \in \bigcup[\nabla]_\infty$. Then the point $[\nabla]_\infty$ is coincident with α. If α is a member of only some, but not all, ultrafilters in $[\nabla]_\infty$, then the complement $-\alpha$ must be a member of the remaining ultrafilters in $[\nabla]_\infty$ so that the point $[\nabla]_\infty$ is coincident with both α and $-\alpha$, and is consequently a border point of α. If, on the other hand, α is a member of every ultrafilter in $[\nabla]_\infty$, and hence a member of $\bigcap[\nabla]_\infty$, then $-\alpha$ does not belong to any ultrafilter in $[\nabla]_\infty$, so that $[\nabla]_\infty$ identifies an interior point of α.

I end this section with some further results concerning filters, which will be needed later.

LEMMA 2.7. *Let* Φ *be a limited proper filter on* Ω *and* $\alpha \infty \Phi$. *Then there is a limited ultrafilter* ∇ *such that* (i) $\alpha \in \nabla$ *and* (ii) $\nabla \infty \Phi$.

Proof. Define a filter Φ' as follows:
$\gamma \in \Phi'$ if and only if (i) $\gamma \infty \beta$, for every $\beta \in \Phi$ and
(ii) $(\alpha \wedge -\gamma) \not\infty \beta$, for some $\beta \in \Phi$.

Φ' is a proper filter with α one of its elements. The only characteristic that is not immediately evident is that whenever γ_1 and γ_2 are both members of Φ' then so is $\gamma_1 \wedge \gamma$. Suppose then that $\gamma_1, \gamma_2 \in \Phi'$. Then $(\alpha \wedge -\gamma_1) \not\infty \beta_1$ and $(\alpha \wedge -\gamma_2) \not\infty \beta_2$, where $\beta_1, \beta_2 \in \Phi$. $\beta_1 \wedge \beta_2 \in \Phi$, as Φ is a filter. So, $(\alpha \wedge -\gamma_1) \not\infty (\beta_1 \wedge \beta_2)$ and $(\alpha \wedge -\gamma_2) \not\infty (\beta_1 \wedge \beta_2)$ by constraint A4. Hence $(\alpha \wedge -(\gamma_1 \wedge \gamma_2)) \not\infty (\beta_1 \wedge \beta_2)$ by A5. Suppose now that there is an element β of Φ such that $(\gamma_1 \wedge \gamma_2) \not\infty \beta$. Then $(\gamma_1 \wedge \gamma_2) \not\infty (\beta \wedge \beta_1 \wedge \beta_2)$ by A4 and $(\alpha \wedge -(\gamma_1 \wedge \gamma_2)) \not\infty (\beta \wedge \beta_1 \wedge \beta_2)$. But then, since $\alpha \leq [(\alpha \wedge -(\gamma_1 \wedge \gamma_2)) \vee (\gamma_1 \wedge \gamma_2)]$, $\alpha \not\infty (\beta \wedge \beta_1 \wedge \beta_2)$; hence $\alpha \notin \Phi'$, which is clearly false. So, $(\gamma_1 \wedge \gamma_2) \infty \beta$ for every element β of Φ, and so $\gamma_1 \wedge \gamma_2 \in \Phi'$. By the definition of Φ', $\Phi' \infty \Phi$. Since Φ is limited, there is a limited region $\beta_0 \in \Phi$. $\alpha \infty \beta_0$ by assumption. Hence there is a limited $\alpha' \leq \alpha$ such that $\alpha' \infty \beta_0$ and $(\alpha \wedge -\alpha') \not\infty \beta_0$, this by Lemma 1.6. $\alpha' \vee \beta_0$ is limited by constraint A8 and $\alpha' \vee \beta_0 \infty \beta$ for every $\beta \in \Phi$. $\alpha \wedge -(\alpha' \vee \beta_0) = \alpha \wedge -\alpha' \wedge -\beta_0$, and this is not connected with β_0. So $(\alpha' \vee \beta_0) \in \Phi'$. Hence Φ' is limited.

Every proper filter can be refined to an ultrafilter.[10] Let ∇ be such a refinement of Φ'. Then ∇ is limited and $\alpha \in \nabla$. Also $\nabla \infty \Phi$. For suppose not. Then there are elements γ of ∇, and β of Φ such that $\gamma \not\infty \beta$. Then $\beta \leq -\gamma$ by Lemma 1.2, hence $-\gamma \in \Phi$, and so $-\gamma \infty \beta'$ for every element β' of Φ by Lemma 1.1. Also $\alpha \wedge -(-\gamma) \not\infty \beta$ by A4. Consequently, $-\gamma \in \Phi'$, and hence $-\gamma \in \nabla$, contradicting the assumption that $\gamma \in \nabla$. So $\nabla \infty \Phi$. □

LEMMA 2.8. *If $\alpha \infty \beta$ then there are limited ultrafilters ∇_α, ∇_β such that $\alpha \in \nabla_\alpha$, $\beta \in \nabla_\beta$ and $\nabla_\alpha \infty \nabla_\beta$.*

Proof. Assume $\alpha \infty \beta$. Then by A9 and A3 there is a limited non-null region $\beta' \leq \beta$ such that $\alpha \infty \beta'$. Since $\alpha \infty \beta'$, $\Phi = \{\gamma \mid \beta' \leq \gamma\}$ is a proper filter and limited, and $\alpha \infty \Phi$. So there exists by Lemma 2.7 a limited ultrafilter ∇_α such that $\alpha \in \nabla_\alpha$ and $\nabla_\alpha \infty \Phi$, and hence $\nabla_\alpha \infty \beta$. And so, by Lemma 2.7 again, there exists a limited ultrafilter ∇_β with $\beta \in \nabla_\beta$ and $\nabla_\alpha \infty \nabla_\beta$. □

Given that by A2 $\alpha \infty \alpha$ when $\alpha \not\leq 0$, Lemma 2.8 has the corollary that for any region α, other than 0, there is a point coincident with it. And two regions α and β are connected with one another if and only if there is a point coincident with both. This will be shown in Section 4 (Theorem 4.7(a)).

3. Alternative Characterisations of Points

While points have been taken to be equivalence classes under ∞ of ultrafilters, there are of course other ways of characterising points in terms of regions. One alternative that suggests itself is to take instead of the equivalence class $[\nabla]_\infty$ its intersection, i.e. the set of regions $\bigcap[\nabla]_\infty$, another to take the union $\bigcup[\nabla]_\infty$. $\bigcap[\nabla]_\infty$ is obviously a filter. In the light of the remarks toward the end of the previous section the two alternatives amount to identifying a point by means of the regions it is an interior point of and identifying it by means of the regions it is coincident with.

LEMMA 3.1. (a) $\alpha \in \bigcap[\nabla]_\infty$ *if and only if* $-\alpha \notin \bigcup[\nabla]_\infty$;
(b) *If* $\alpha \not\infty \beta$ *and* $\alpha \in \bigcup[\nabla]_\infty$, *then* $-\beta \in \bigcap[\nabla]_\infty$. (*If* $\alpha \ll \gamma$ *and* $\alpha \in \bigcup[\nabla]_\infty$, *then* $\gamma \in \bigcap[\nabla]_\infty$.)

Proof. (a) By virtue of the fact that any ultrafilter on Ω has exactly one of α and $-\alpha$ as an element. (b) Suppose $\alpha \in \nabla$ and $\alpha \not\infty \beta$. Then if $\nabla' \infty \nabla$, $\beta \notin \nabla'$. Hence $-\beta \in \nabla'$, and so $-\beta \in \bigcap[\nabla]_\infty$. □

LEMMA 3.2. *If* $\bigcap[\nabla]_\infty = \bigcap[\nabla']_\infty$, *then* $\nabla \infty \nabla'$.

Proof. Assume $\nabla \not\infty \nabla'$. Then there are regions $\alpha \in \nabla$ and $\beta \in \nabla'$ with $\alpha \not\infty \beta$. Since $\alpha \in \bigcup[\nabla]_\infty$, $-\beta \in \bigcap[\nabla]_\infty$ by Lemma 3.1(b). And since $-\beta \notin \nabla'$, $-\beta \notin \bigcap[\nabla']_\infty$. So $\bigcap[\nabla]_\infty \neq \bigcap[\nabla']_\infty$. □

It then follows that

LEMMA 3.3. $\nabla \infty \nabla'$ *if and only if* $[\nabla]_\infty = [\nabla']_\infty$ *if and only if* $\bigcap[\nabla]_\infty = \bigcap[\nabla']_\infty$.

LEMMA 3.4. *Let* ∇ *be a limited ultrafilter. Then* $\nabla' \infty \nabla$ *if* $\nabla' \infty \bigcap[\nabla]_\infty$.

Proof. Suppose $\nabla \not\infty \nabla'$. Then by Lemma 2.5 there are a limited region $\alpha \in \nabla$ and a region $\beta \in \nabla'$ such that $\alpha \not\infty \beta$. By Lemma 1.4 there is then a region γ with $\alpha \ll \gamma$ and $\beta \not\infty \gamma$. Hence $\gamma \in \bigcap[\nabla]_\infty$ by Lemma 3.1(b). Hence $\nabla' \not\infty \bigcap[\nabla]_\infty$. □

Now the sets of regions that are intersections of equivalence classes under ∞ of limited ultrafilters can be specified directly, namely as maximal proper, limited, and contracting filters on Ω. But first, the notion of a *contracting* filter needs to be defined.

DEFINITION 3.1. A filter Φ on Ω is said to be *contracting* if for every region α in Φ there is a region β in Φ with $\beta \ll \alpha$.

LEMMA 3.5. *Let ∇ be a limited ultrafilter on Ω. Then $\bigcap[\nabla]_\infty$ is a maximal proper, limited, and contracting filter.*

Proof. (i) $\bigcap[\nabla]_\infty$ is a proper filter, since ultrafilters are proper filters. (ii) Since ∇ is limited, there is a limited region α in ∇. By Lemma 1.5(a) there is a limited region β with $\alpha \ll \beta$. So $\beta \in \bigcap[\nabla]_\infty$ by Lemma 3.1(b) and $\bigcap[\nabla]_\infty$ is limited. (iii) Suppose that $\alpha \in \bigcap[\nabla]_\infty$, but that, for every region β in $\bigcap[\nabla]_\infty$, $-\alpha \infty \beta$, i.e. $-\alpha \infty \bigcap[\nabla]_\infty$. By Lemma 2.7 there is then an ultrafilter ∇' with $-\alpha \in \nabla'$ and $\nabla' \infty \bigcap[\nabla]_\infty$. Hence $\nabla' \in [\nabla]_\infty$ by Lemma 3.4, so that $\alpha \in \nabla'$, q.e.a. So there exists a region β in $\bigcap[\nabla]_\infty$ with $\beta \ll \alpha$, and $\bigcap[\nabla]_\infty$ is contracting. (iv) Suppose $\bigcap[\nabla]_\infty$ is a proper subset of a proper, contracting filter Φ, and let α be in Φ but not in $\bigcap[\nabla]_\infty$. Then $-\alpha \in \bigcup[\nabla]_\infty$ by Lemma 3.1(a). Since Φ is contracting there is a region β in Φ with $\beta \not\infty -\alpha$. By Lemma 3.1(b) $-\beta \in \bigcap[\nabla]_\infty$. Hence $-\beta$ is a member of Φ, which is impossible, since Φ was supposed to be a proper filter. Hence $\bigcap[\nabla]_\infty$ is maximal. □

LEMMA 3.6. *Let Φ be a maximal proper, limited, and contracting filter on Ω. Then there is a limited ultrafilter ∇ such that $\Phi = \bigcap[\nabla]_\infty$.*

Proof. Since Φ is a limited filter there is a limited ultrafilter ∇ with $\Phi \subseteq \nabla$. Suppose that $\nabla \infty \nabla'$ and $\alpha \in \Phi$, and hence $\alpha \in \nabla$. If $\alpha \notin \nabla'$, $-\alpha \in \nabla'$. Since Φ is contracting, there is a region β in Φ, and hence in ∇, with $\beta \not\infty -\alpha$. But this contradicts the supposition that $\nabla \infty \nabla'$. So if $\nabla \infty \nabla'$, $\Phi \subseteq \nabla'$. Hence $\Phi \subseteq \bigcap[\nabla]_\infty$. But since Φ is maximal, $\Phi = \bigcap[\nabla]_\infty$ by Lemma 3.5. □

It is clear from the proof that, for any limited ultrafilter ∇ extending Φ, $\Phi = \bigcap[\nabla]_\infty$.

In virtue of Lemmas 3.3, 3.5, and 3.6 points can equivalently be characterised as maximal proper, limited, and contracting filters on Ω. What this amounts to in the light of the remarks toward the end of Section 2 is to identify a point by the totality of regions in whose *interior* it lies.

The second alternative characterisation of points employs the sets $\bigcup[\nabla]_\infty$ for limited ∇. These can also be described directly as *coincidence sets*.

DEFINITION 3.2. A set Σ of regions is a *coincidence set* if it meets the following four conditions:

(i) If α and β are members of Σ then $\alpha \infty \beta$;
(ii) If $\alpha \vee \beta \in \Sigma$, then $\alpha \in \Sigma$ or $\beta \in \Sigma$;
(iii) If $\alpha \infty \beta$ for every β in Σ, then $\alpha \in \Sigma$;

(iv) At least one member of Σ is limited.

The remainder of the section is concerned with showing that the sets $\bigcup[\nabla]_\infty$ are just the coincidence sets.

LEMMA 3.7. *If ∇ is a limited ultrafilter on Ω, then $\bigcup[\nabla]_\infty$ is a coincidence set.*

Proof. Clauses (i) and (iv) of Definition 3.2 are obviously met. As for (ii), suppose that $\alpha \vee \beta \in \bigcup[\nabla]_\infty$. Then $\alpha \vee \beta \in \nabla'$, where ∇' is an ultrafilter on Ω that is collocated with ∇. Consequently $\alpha \in \nabla'$ or $\beta \in \nabla'$, and so $\alpha \in \Sigma$ or $\beta \in \Sigma$. And as for (iii), suppose that $\alpha \infty \beta$ for every β in $\bigcup[\nabla]_\infty$. Then $\alpha \infty \nabla$; and so there exists by Lemma 2.7 and ultrafilter ∇' on Ω with $\alpha \in \nabla'$ and $\nabla' \infty \nabla$. Hence $\alpha \in \bigcup[\nabla]_\infty$. □

LEMMA 3.8. *Let Σ be a coincidence set and let Φ_Σ be the set $\{\alpha \mid -\alpha \notin \Sigma\}$. Then*

(a) *Φ_Σ is a proper filter on Ω and a subset of Σ;*
(b) *$\alpha \in \Sigma$ if and only if $\alpha \infty \beta$ for every β in Φ_Σ.*

Proof. (a) Assume *first* that $\alpha \leq \beta$ and $\alpha \in \Phi_\Sigma$, i.e. $-\alpha \notin \Sigma$. Then by clause (iii) of Definition 3.2 there is a region γ in Σ with $-\alpha \not\infty \gamma$. Hence $-\beta \not\infty \gamma$ by A1 and A4, hence $-\beta \notin \Sigma$, and so $\beta \in \Phi_\Sigma$. Suppose *next* that $\alpha \in \Phi_\Sigma$ and $\beta \in \Phi_\Sigma$, i.e. $-\alpha \notin \Sigma$ and $-\beta \notin \Sigma$. Then $-\alpha \vee -\beta \notin \Sigma$ by clause (ii) and so $\alpha \wedge \beta \in \Phi$. Since $1 \in \Sigma$, $0 \notin \Phi_\Sigma$, and so Φ_Σ is a proper filter. Finally, if $\alpha \in \Phi_\Sigma$, $-\alpha \notin \Sigma$, and hence $\alpha \in \Sigma$ by (ii). So Φ_Σ is a subset of Σ. (b) Suppose $\alpha \notin \Sigma$. Then there is a region γ in Σ such that $\alpha \not\infty \gamma$. Hence there exists by Lemma 1.4 a region β with $\alpha \not\infty \beta$ and $-\beta \not\infty \gamma$. Hence $-\beta \notin \Sigma$, and so $\beta \in \Phi_\Sigma$. So if $\alpha \infty \beta$ for every β in Φ_Σ, then $\alpha \in \Sigma$. The converse follows from (a) and clause (i). □

LEMMA 3.9. *Let Σ be a coincidence set. Then $\Sigma = \bigcup[\nabla]_\infty$ for some limited ultrafilter ∇.*

Proof. Let Φ_Σ be as before. Let ∇ be any ultrafilter that is a refinement of Φ_Σ. Since $\Phi_\Sigma \subseteq \nabla$ and ∇ is a proper filter, $\alpha \wedge \beta \notin 0$ for every $\alpha \in \nabla$ and $\beta \in \Phi_\Sigma$. Hence $\nabla \subseteq \Sigma$ by Lemma 3.8(b). Hence by clauses (i) and (iv) of Definition 3.2 there is a limited region α with $\alpha \infty \nabla$; and so ∇ is limited by Lemma 2.1. Now assume *first* that $\nabla' \infty \nabla$ and $\alpha \in \nabla'$. Then $\alpha \infty \beta$ for every $\beta \in \Phi_\Sigma$ and hence $\alpha \in \Sigma$ by Lemma 3.8(b). Hence $\bigcup[\nabla]_\infty \subseteq \Sigma$. Assume *next* that $\alpha \in \Sigma$. Then $\alpha \infty \nabla$ by condition (i). So there is an ultrafilter ∇' with $\alpha \in \nabla'$ and $\nabla' \infty \nabla$ by Lemma 2.7. Hence $\alpha \in \bigcup[\nabla]_\infty$. So $\Sigma \subseteq \bigcup[\nabla]_\infty$. □

So points can be characterised by coincidence sets. And any such set is the totality of all those regions that a given point is coincident with.

Note that the regions in $\bigcap[\nabla]_\infty$ and $\bigcup[\nabla]_\infty$ are not necessarily convex or coherent. The characterisation of points by sets of convex regions will be considered in Section 6.

4. Extracting the Point-Based Topology

Having identified ways of characterising points in a region-based topology $\mathcal{R}$, one can turn to the question: which topology (in the sense of point-based topology) do these points have? The topology of a set of points is completely described by its closed subsets. So, given the non-empty set of points $P_\mathcal{R}$ of the region-based topology $\mathcal{R} = \langle \Omega, \infty, \Delta \rangle$, identifying the family $\mathbf{C}_\mathcal{R}$ of closed sets of points as determined by $\mathcal{R}$ is all that is needed to yield a topological space $\mathcal{P}_\mathcal{R} = \langle P_\mathcal{R}, \mathbf{C}_\mathcal{R} \rangle$ in the usual sense of point set topology.

The characterisation of the closed subsets of $P_\mathcal{R}$ is fairly straightforward. To any region in Ω two sets of points can be seen to correspond; namely, *firstly*, the set of interior points; and, *secondly*, the set of all points coincident with the region, be they interior or boundary points. Consequently, I define two functions $\mathbf{I}$ and $\mathbf{C}$ from regions to sets of points. ∇ and ∇' are to be limited ultrafilters on Ω. Indeed, from now on it will be assumed that the ultrafilter mentioned in '$[\nabla]_\infty$' is limited, so that '$[\nabla]_\infty$' always designates a point.

DEFINITION 4.1. (a) $\mathbf{I}(\alpha) = \{[\nabla]_\infty \mid \alpha \in \bigcap[\nabla]_\infty\}$;
(b) $\mathbf{C}(\alpha) = \{[\nabla]_\infty \mid \alpha \in \bigcup[\nabla]_\infty\}$.

This means that

$$[\nabla]_\infty \in \mathbf{I}(\alpha) \text{ if and only if } \alpha \in \nabla' \text{ for every } \nabla' \text{ in} [\nabla]_\infty$$

and

$$[\nabla]_\infty \in \mathbf{C}(\alpha) \text{ if and only if } \alpha \in \nabla' \text{ for some } \nabla' \text{ in} [\nabla]_\infty.$$

If points are identified by maximally contracting, limited filters Φ, then $\mathbf{I}(\alpha)$ comes to

$$\{\Phi \mid \alpha \in \Phi\},$$

and $\mathbf{C}(\alpha)$ comes to

$$\{\Phi \mid -\alpha \notin \Phi\},$$

this by Lemma 3.1(a).

Obviously, $\mathbf{I}(\alpha)$ should be an open set, $\mathbf{C}(\alpha)$ a closed one. It turns out that the sets $\mathbf{I}(\alpha)$ and $\mathbf{C}(\alpha)$ are uniquely associated with the region α. The following lemmas are concerned with this and other relevant results.

LEMMA 4.1. $\mathbf{I}(\alpha)$ *is* $\backslash\mathbf{C}(-\alpha)$, *the complement of* $\mathbf{C}(-\alpha)$ *in* $P_{\mathcal{R}}$.
Proof by Lemma 3.1(a).

LEMMA 4.2. (a) $\mathbf{I}(\alpha) \subseteq \mathbf{C}(\alpha)$;
(b) *If* $\alpha \leq \beta$, *then* $\mathbf{I}(\alpha) \subseteq \mathbf{I}(\beta)$ *and* $\mathbf{C}(\alpha) \subseteq \mathbf{C}(\beta)$;
(c) *If* $\alpha \wedge \beta = 0$, *then* $\mathbf{I}(\alpha) \cap \mathbf{C}(\beta) = \varnothing$;
(d) *If* $\alpha \ll \beta$, *then* $\mathbf{C}(\alpha) \subseteq \mathbf{I}(\beta)$.

Proof. (a) and (b) are immediate consequences of the definition of $\mathbf{I}(\alpha)$ and $\mathbf{C}(\alpha)$ and the characteristics of filters. (c) by (b) and Lemma 4.1. (d) Assume $\alpha \ll \beta$ and $[\nabla]_\infty \in \mathbf{C}(\alpha)$, i.e. $\alpha \in \bigcup[\nabla]_\infty$. Then $\beta \in \bigcap[\nabla]_\infty$ by Lemma 3.1(b); i.e. $[\nabla]_\infty \in \mathbf{I}(\beta)$. □

LEMMA 4.3. (a) $\mathbf{I}(\alpha \wedge \beta) = \mathbf{I}(\alpha) \cap \mathbf{I}(\beta)$;
(b) $\mathbf{C}(\alpha \vee \beta) = \mathbf{C}(\alpha) \cup \mathbf{C}(\beta)$.

Proof. Let ∇ be any limited ultrafilter. Then (a) $[\nabla]_\infty \in \mathbf{I}(\alpha \wedge \beta)$ if and only if $\alpha \wedge \beta \in \bigcap[\nabla]_\infty$ if and only if, by Lemma 3.5, $\alpha \in \bigcap[\nabla]_\infty$ and $\beta \in \bigcap[\nabla]_\infty$ if and only if $[\nabla]_\infty \in \mathbf{I}(\alpha)$ and $[\nabla]_\infty \in \mathbf{I}(\beta)$. (b) $[\nabla]_\infty \in \mathbf{C}(\alpha \vee \beta)$ if and only if $\alpha \vee \beta \in \bigcup[\nabla]_\infty$ if and only if, by Lemma 3.7, $\alpha \in \bigcup[\nabla]_\infty$ or $\beta \in \bigcup[\nabla]_\infty$ if and only if $[\nabla]_\infty \in \mathbf{C}(\alpha)$ or $[\nabla]_\infty \in \mathbf{C}(\beta)$. □

LEMMA 4.4. (a) $\mathbf{I}(0) = \varnothing = \mathbf{C}(0)$;
(b) *If* $\mathbf{C}(\alpha) = \varnothing$, *then* $\alpha = 0$;
(c) *If* $\mathbf{I}(\alpha) = \varnothing$, *then* $\alpha = 0$;
(d) *If* $\mathbf{I}(1) = P_{\mathcal{R}} = \mathbf{C}(1)$.

Proof. (a) There is no ultrafilter ∇ with $0 \in \nabla$, since ultrafilters are proper filters. (b) Suppose $\alpha \neq 0$. Then by Lemma 1.3(b) there exists a limited non-null region β with $\beta \leq \alpha$. Any ultrafilter ∇ that is a refinement of the proper filter generated by β is limited. And since $\beta \in \nabla$, $[\nabla]_\infty \in \mathbf{C}(\alpha)$, so that $\mathbf{C}(\alpha) \neq \varnothing$. (c) Assume $\alpha \neq 0$. By Lemma 1.3(a) there exists then a non-null region β with $\beta \ll \alpha$. Hence $\mathbf{C}(\beta) \subseteq \mathbf{I}(\alpha)$ by Lemma 4.2(b). Since $\beta \neq 0$, $\mathbf{C}(\beta) \neq \varnothing$ by (b). So $\mathbf{I}(\alpha) \neq \varnothing$. (d) 1 belongs to every ultrafilter on Ω. □

LEMMA 4.5. (a) *If* $\mathbf{I}(\alpha) \subseteq \mathbf{C}(\beta)$ *then* $\alpha \leq \beta$;
(b) *If* $\mathbf{I}(\alpha) \subseteq \mathbf{I}(\beta)$ *then* $\alpha \leq \beta$;
(c) *If* $\mathbf{C}(\alpha) \subseteq \mathbf{C}(\beta)$ *then* $\alpha \leq \beta$;
(d) *If* $\mathbf{C}(\alpha) \subseteq \mathbf{I}(\beta)$ *then* $\alpha \leq \beta$.

Proof. Suppose not $\alpha \leq \beta$. Then $\alpha \wedge -\beta \neq 0$. Hence $\mathsf{I}(\alpha \wedge -\beta) \neq \varnothing$ by Lemma 4.4(c) and so $\mathsf{I}(\alpha) \cap \mathsf{I}(-\beta) \neq \varnothing$ by Lemma 4.3(a). But $\mathsf{I}(-\beta) = \backslash\mathsf{C}(\beta)$ by Lemma 4.1. So (a). (b), (c), and (d) by (a) and Lemma 4.2(a). □

THEOREM 4.1. (a) $\mathsf{I}(\alpha) = \mathsf{I}(\beta)$ *if and only if* $\alpha = \beta$;
(b) $\mathsf{C}(\alpha) = \mathsf{C}(\beta)$ *if and only if* $\alpha = \beta$.
Proof by Lemma 4.5.

So the sets $\mathsf{C}(\alpha)$, with α ranging over the regions in Ω correspond one-to-one to those regions; and they are some of the closed sets of $\mathcal{P}_{\mathcal{R}}$. But not every closed set is $\mathsf{C}(\alpha)$ for some region α of $\mathcal{R}$. However, the family $\{\mathsf{C}(\alpha) \mid \alpha \in \Omega\}$ can serve as a *basis* of the topology on $P_{\mathcal{R}}$. A closed (open) basis of a given point-based topology is a family $\mathbf{B}$ of sets with the feature that a set of points is closed (open) if and only if it is the intersection (union) of a non-empty subset of $\mathbf{B}$. So if $\mathbf{B}$ is a closed basis then every closed set C equals $\bigcap\{A \mid A \in \mathbf{B}$ and $C \subseteq A\}$, and if $\mathbf{B}$ is an open basis then every open set O equals $\bigcup\{A \mid A \in \mathbf{B}$ and $A \subseteq O\}$.

A sufficient condition for a family $\mathbf{B}$ of subsets of space P to be a closed basis of some topology or other on P is that (i) $P \in \mathbf{B}$, (ii) $\bigcap\mathbf{B} = \varnothing$, and (iii) $B_1 \cup B_2 \in \mathbf{B}$ when $B_1, B_2 \in \mathbf{B}$. $\{\mathsf{C}(\alpha) \mid \alpha \in \Omega\}$ meets these conditions.

THEOREM 4.2. *The class* $\mathbf{B} = \{\mathsf{C}(\alpha) \mid \alpha \in \Omega\}$ *is the basis of a topology on* $P_{\mathcal{R}}$.

Proof. (i) $P_{\mathcal{R}} \in \mathbf{B}$, since $P_{\mathcal{R}} = \mathsf{C}(1)$ by Lemma 4.4(d). (ii) $\bigcap\mathbf{B} = \varnothing$ by Lemma 4.4(a). (iii) If $B_1, B_2 \in \mathbf{B}$, then $B_1 \cup B_2 \in \mathbf{B}$ by Lemma 4.3(b). □

This then allows us to stipulate that the closed subsets of $P_{\mathcal{R}}$ are just those sets that are the intersections of non-empty subfamilies of $\{\mathsf{C}(\alpha) \mid \alpha \in \Omega\}$. As already indicated, this class (of the closed sets in $P_{\mathcal{R}}$) will be referred to as $\mathbf{C}_{\mathcal{R}}$. As a consequence of this definition the open sets of $P_{\mathcal{R}}$ are then the unions of non-empty subfamilies of $\{\mathsf{I}(\alpha) \mid \alpha \in \Omega\}$.

DEFINITION 4.2. Let $\mathcal{R}$ be a region-based topology, $P_{\mathcal{R}} = \mathsf{C}(1)$ the set of points defined on $\mathcal{R}$, $\mathbf{C}_{\mathcal{R}}$ the family of intersections of non-empty subfamilies of $\{\mathsf{C}(\alpha) \mid \alpha \in \Omega\}$. Then the *point-based topology* $P_{\mathcal{R}}$ is the structure $\langle P_{\mathcal{R}}, \mathbf{C}_{\mathcal{R}}\rangle$.

Such a point-based structure $P_{\mathcal{R}}$ corresponds to just one (up to isomorphisms) region-based topology as will be shown below (Theorem 4.5). First some auxiliary results:

LEMMA 4.6. *Let C be a closed set and O an open set of $P_{\mathcal{R}}$. Then*

(a) $C = \bigcap\{\mathbf{C}(\alpha) \mid C \subseteq \mathbf{C}(\alpha)\}$ *and*
(b) $O = \bigcup\{\mathbf{I}(\alpha) \mid \mathbf{I}(\alpha) \subseteq O\}$.

Proof by the definition of the topology. □

The next lemma establishes that, for any region α, $\mathbf{C}(\alpha)$ is regular-closed and $\mathbf{I}(\alpha)$ is regular-open. Cl and In are the closure and interior operations, respectively.

DEFINITION 4.3. Let A be a subset of a topological point-space. Then A is *regular-closed* if $\mathrm{Cl}\,\mathrm{In}\,A = A$, and A is *regular-open* if $\mathrm{In}\,\mathrm{Cl}\,A = A$.

LEMMA 4.7. (a) $\mathrm{Cl}\,\mathbf{I}(\alpha) = \mathbf{C}(\alpha)$;
(b) $\mathrm{In}\,\mathbf{C}(\alpha) = \mathbf{I}(\alpha)$;
(c) $\mathbf{C}(\alpha)$ *is regular-closed and* $\mathbf{I}(\alpha)$ *is regular-open.*

Proof. (a) $\mathrm{Cl}\,\mathbf{I}(\alpha)$ is the smallest closed set containing $\mathbf{I}(\alpha)$, i.e. $\mathrm{Cl}\,\mathbf{I}(\alpha)$ equals $\bigcap\{C \mid \mathbf{I}(\alpha) \subseteq C\}$, which equals by Lemma 4.6(a) $\bigcap\{\mathbf{C}(\beta) \mid \mathbf{I}(\alpha) \subseteq \mathbf{C}(\beta)\}$, which equals by Lemmas 4.5(a) and 4.2(b) $\bigcap\{\mathbf{C}(\beta) \mid \mathbf{C}(\alpha) \subseteq \mathbf{C}(\beta)\}$, which equals $\mathbf{C}(\alpha)$. (b) Analogously. (c) By (a) and (b). □

THEOREM 4.3. *Let the Boolean algebra Ω in $\mathcal{R} = \langle \Omega, \infty, \Delta \rangle$ be complete, let C be any regular-closed set of $P_{\mathcal{R}}$ and let O be any regular-open set of $P_{\mathcal{R}}$. Then*

(a) *C equals $\mathbf{C}(\alpha)$ for some region α of Ω;*
(b) *O equals $\mathbf{I}(\alpha)$ for some region α of Ω.*

Proof. (a) Let Γ be $\{\gamma \mid C \subseteq \mathbf{C}(\gamma)\}$. Then by Lemma 4.6.

(1) $C = \bigcap\{\mathbf{C}(\gamma) \mid \gamma \in \Gamma\}$ and
(2) $\mathrm{In}\,C = \bigcup\{\mathbf{I}(\beta) \mid \mathbf{I}(\beta) \subseteq C\}$.

Assume $\mathbf{I}(\beta) \subseteq C$. Then $\mathbf{I}(\beta) \subseteq \mathbf{C}(\gamma)$ for every γ in Γ by (1) and hence $\beta \leq \gamma$ for every such γ by Lemma 4.5(a). But then $\beta \leq \Lambda\Gamma$, and so $\mathbf{I}(\beta) \subseteq \mathbf{C}(\Lambda\Gamma)$ by Lemma 4.2. So if $\mathbf{I}(\beta) \subseteq C$, then $\mathbf{I}(\beta) \subseteq \mathbf{C}(\Lambda\Gamma)$. But then $\bigcup\{\mathbf{I}(\beta) \mid \mathbf{I}(\beta) \subseteq C\} \subseteq \mathbf{C}(\Lambda\Gamma)$, i.e., by virtue of (2), $\mathrm{In}\,C \subseteq \mathbf{C}(\Lambda\Gamma)$. So $C \subseteq \mathbf{C}(\Lambda\Gamma)$, since C is regular-closed and $\mathbf{C}(\Lambda\Gamma)$ is closed. But $\mathbf{C}(\Lambda\Gamma) \subseteq \bigcap\{\mathbf{C}(\gamma) \mid \gamma \in \Gamma\}$ by Lemma 4.2(b), and hence $C = \mathbf{C}(\Lambda\Gamma)$ by (1). (b) Analogously. □

For simplicity's sake I shall call a region-based topology $\mathcal{R} = \langle \Omega, \infty, \Delta \rangle$ itself *complete* when the Boolean algebra Ω is complete. So, given Theorem 4.1 and Lemma 4.7(c), we have

THEOREM 4.4. *If $\mathcal{R}$ is complete, then* $\mathbf{C}$ *constitutes a one-to-one mapping of the regions of $\mathcal{R}$ onto the regular-closed sets of $\mathcal{P}_{\mathcal{R}}$.*

Assume that $\mathcal{R}_1 = \langle \Omega_1, \infty_1, \Delta_1 \rangle$ and $\mathcal{R}_2 = \langle \Omega_2, \infty_2, \Delta_2 \rangle$ are isomorphic structures. Then by virtue of the definition of *point* and Theorems 4.1 and 4.2 there is a one-to-one correspondence between the spaces $P_{\mathcal{R}_1}$ and $P_{\mathcal{R}_2}$ that correlates the closed sets of $\mathcal{P}_{\mathcal{R}_1} = \langle P_{\mathcal{R}_1}, \mathbf{C}_{\mathcal{R}_1} \rangle$ with the closed sets of $\mathcal{P}_{\mathcal{R}_2} = \langle P_{\mathcal{R}_2}, \mathbf{C}_{\mathcal{R}_2} \rangle$, i.e. $\mathcal{P}_{\mathcal{R}_1}$ and $\mathcal{P}_{\mathcal{R}_2}$ are structurally the same (they are *homeomorphic*). The converse is not true; different region-based topologies may give rise to structurally identical point-based topologies. However, if we confine our attention to region-based topologies with *complete* Boolean algebras of regions then Theorem 4.4 guarantees that if $\mathcal{P}_{\mathcal{R}_1}$ and $\mathcal{P}_{\mathcal{R}_2}$ are homeomorphic then $\mathcal{R}_1$ and $\mathcal{R}_2$ are isomorphic. So,

THEOREM 4.5. *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be complete region-based topologies. Then $\mathcal{R}_1$ and $\mathcal{R}_2$ are isomorphic if and only if $\mathcal{P}_{\mathcal{R}_1}$ and $\mathcal{P}_{\mathcal{R}_2}$ are homeomorphic.*[11]

Having established the one-to-one correspondence between region-based topologies and the point-based topologies they give rise to in virtue of the chosen characterisation of *point* and *closed set of points*, the question arises what kind of point-based topology is obtained in this way. But before this can be addressed it has to be clarified which properties of, relations among, and functions on the regular-closed sets of $\mathcal{P}_{\mathcal{R}}$ correspond to the properties of, relations among, and functions on the regions of $\mathcal{R}$ whose counterparts by the function $\mathbf{C}$ they are. First the Boolean functions and relations.

THEOREM 4.6. (a) $\alpha \leq \beta$ *if and only if* $\mathbf{C}(\alpha) \subseteq \mathbf{C}(\beta)$ *if and only if* $\mathbf{I}(\alpha) \subseteq \mathbf{I}(\beta)$;
(b) $\mathbf{C}(-\alpha) = \backslash \operatorname{In} \mathbf{C}(\alpha) = \operatorname{Cl} \backslash \mathbf{C}(\alpha)$;
(c) $\mathbf{C}(\alpha \vee \beta) = \mathbf{C}(\alpha) \cup \mathbf{C}(\beta)$;
(d) $\mathbf{C}(\alpha \wedge \beta) = \operatorname{Cl}(\operatorname{In} \mathbf{C}(\alpha) \cap \operatorname{In} \mathbf{C}(\beta))$;
(e) *If* $\mathrm{V}\Gamma$ *exists,* $\mathbf{C}(\mathrm{V}\Gamma) = \operatorname{Cl} \bigcup\{\mathbf{C}(\alpha) \mid \alpha \in \Gamma\}$;
(f) *If* $\Lambda\Gamma$ *exists,* $\mathbf{C}(\Lambda\Gamma) = \operatorname{Cl} \operatorname{In} \bigcap\{\mathbf{C}(\alpha) \mid \alpha \in \Gamma\}$.

Proof. (a) By Lemma 4.2(b) and Lemma 4.5. (b) By Lemmas 4.1 and 4.7. (c) By Lemma 4.3. (d) By (b) and (c) and the identity $\alpha \wedge \beta = -(\alpha \vee -\beta)$. (e) Since $\mathbf{C}(\alpha) \subseteq \mathbf{C}(\mathrm{V}\Gamma)$ for every α in Γ by Lemma 4.2(b), $\bigcup\{\mathbf{C}(\alpha) \mid \alpha \in \Gamma\} \subseteq \mathbf{C}(\mathrm{V}\Gamma)$ and, as $\mathbf{C}(\mathrm{V}\Gamma)$ is closed, $\operatorname{Cl} \bigcup\{\mathbf{C}(\alpha) \mid \alpha \in \Gamma\} \subseteq \mathbf{C}(\mathrm{V}\Gamma)$. And if $\bigcup\{\mathbf{C}(\alpha) \mid \alpha \in \Gamma\} \subseteq \mathbf{C}(\beta)$ then, by Lemmas 4.5(c) and 4.2(b), $\mathbf{C}(\mathrm{V}\Gamma) \subseteq \mathbf{C}(\beta)$. Since $\operatorname{Cl} \bigcup\{\mathbf{C}(\alpha) \mid \alpha \in \Gamma\} = \bigcap\{\mathbf{C}(\beta) \mid \bigcup\{\mathbf{C}(\alpha) \mid \alpha \in \Gamma\} \subseteq \mathbf{C}(\beta)\}$, $\operatorname{Cl} \bigcup\{\mathbf{C}(\alpha) \mid \alpha \in \Gamma\} \supseteq \mathbf{C}(\mathrm{V}\Gamma)$. (f) Since

$\mathbf{I}(\Lambda\Gamma) \subseteq \mathbf{C}(\alpha)$ for every α in Γ by Lemma 4.2(a) and (b), $\mathbf{I}(\Lambda\Gamma) \subseteq \bigcap\{\mathbf{C}(\alpha) \mid \alpha \in \Gamma\}$ and $\mathbf{I}(\Lambda\Gamma) \subseteq \mathrm{In}\bigcap\{\mathbf{C}(\alpha) \mid \alpha \in \Gamma\}$. And if $\mathbf{I}(\beta) \subseteq \bigcap\{\mathbf{C}(\alpha) \mid \alpha \in \Gamma\}$ then, by Lemmas 4.5(a) and 4.2(b), $\mathbf{I}(\beta) \subseteq \mathbf{I}(\Lambda\Gamma)$. So, since $\mathrm{In}\bigcap\{\mathbf{C}(\alpha) \mid \alpha \in \Gamma\} = \bigcup\{\mathbf{I}(\beta) \mid \mathbf{I}(\beta) \subseteq \bigcap\{\mathbf{C}(\alpha) \mid \alpha \in \Gamma\}\}$, $\mathrm{In}\bigcap\{\mathbf{C}(\alpha) \mid \alpha \in \Gamma\} \subseteq \mathbf{I}(\Lambda\Gamma)$. Hence $\mathbf{I}(\Lambda\Gamma) = \mathrm{In}\bigcap\{\mathbf{C}(\alpha) \mid \alpha \in \Gamma\}$. So, by Lemma 4.7(a), $\mathbf{C}(\Lambda\Gamma) = \mathrm{Cl}\,\mathrm{In}\bigcap\{\mathbf{C}(\alpha) \mid \alpha \in \Gamma\}$. □

The null-region corresponds to the empty set and the whole space, i.e. the 1 element of Ω, corresponds to $P_{\mathcal{R}}$, as established by Lemma 4.4. The next theorem deals with the relation of connection.

THEOREM 4.7. (a) *$\alpha \infty \beta$ if and only if $\mathbf{C}(\alpha) \cap \mathbf{C}(\beta) \neq \varnothing$;*
(b) *If $\mathbf{C}(\alpha) \subseteq \mathbf{I}(\beta)$ then $\alpha \ll \beta$.*

Proof. (a) (i) Suppose $\alpha \infty \beta$. Then by Lemma 2.8 there are limited ultrafilters ∇_α and ∇_β such that $\alpha \in \nabla_\alpha$, $\beta \in \nabla_\beta$, and $\nabla_\alpha \infty \nabla_\beta$. Hence $[\nabla_\alpha]_\infty \in \mathbf{C}(\alpha)$ and $[\nabla_\alpha]_\infty \in \mathbf{C}(\beta)$. (ii) Suppose $[\nabla]_\infty \in \mathbf{C}(\alpha)$ and $[\nabla]_\infty \in \mathbf{C}(\beta)$, where ∇ is limited. Then $\alpha \in \bigcup[\nabla]_\infty$ and $\beta \in \bigcup[\nabla]_\infty$. Hence $\alpha \infty \beta$ by Lemma 3.7. (b) By (a) and Lemma 4.1. □

It is now easy to see that in the point-based topology $\mathcal{P}_{\mathcal{R}}$ all singleton sets of points are closed.

LEMMA 4.8. *All 1-point sets in $P_{\mathcal{R}}$ are closed.*

Proof. Let $[\nabla]_\infty$ be a point of $P_{\mathcal{R}}$. Then $\{[\nabla]_\infty\}$ is identical with the closed set $\bigcap\{\mathbf{C}(\alpha) \mid [\nabla]_\infty \in \mathbf{C}(\alpha)\}$. For suppose $[\nabla']_\infty$ is any point other than $[\nabla]_\infty$. Then $\nabla \not\infty \nabla'$; i.e. there are regions $\alpha \in \nabla$ and $\beta \in \nabla'$ with $\alpha \not\infty \beta$. By Definition 4.1 $[\nabla]_\infty \in \mathbf{C}(\alpha)$ and $[\nabla']_\infty \in \mathbf{C}(\beta)$, and by Theorem 4.7(a) $\mathbf{C}(\alpha) \cap \mathbf{C}(\beta) = \varnothing$. So $[\nabla']_\infty \notin \bigcap\{\mathbf{C}(\alpha) \mid [\nabla]_\infty \in \mathbf{C}(\alpha)\}$. □

Next it is shown that the *limited* regions of $\mathcal{R}$ correspond to the *compact* regular-closed sets of $P_{\mathcal{R}}$. This may be somewhat surprising if one thinks of the usual definition of compactness in terms of covers of open sets. But the correspondence is clearer if one uses the equivalent characterisation of *compact* closed sets contained in the following lemma.

LEMMA 4.9. *A closed subset C of a topological space P is compact if and only if for every proper filter* **F** *on P with $C \in$* **F** *there is a point $x \in C$ such that every open set O with $x \in O$ overlaps with every member of* **F**.

The compactness of $\mathbf{C}(\alpha)$, where α is limited, then follows from the correspondence between filters on Ω and filters on $P_{\mathcal{R}}$, and the definition of points in terms of limited ultrafilters on Ω.

LEMMA 4.10. (a) *Let* $[\nabla]_\infty$ *be a point and* O *an open set such that* $[\nabla]_\infty \in O$. *Then there is a limited region* α *such that* $[\nabla]_\infty \in \mathbf{I}(\alpha)$ *and* $\mathbf{C}(\alpha) \subseteq O$;
(b) *For any point* $[\nabla]_\infty$ *there is a limited region* α *such that* $[\nabla]_\infty \in \mathbf{I}(\alpha)$.

Proof. (a) By Lemma 4.6(b) there is a region β with $\mathbf{I}(\beta) \subseteq O$ and $[\nabla]_\infty \in \mathbf{I}(\beta)$, i.e. $\beta \in \bigcap[\nabla]_\infty$ by Definition 4.1(a). By Lemma 3.5 there exists a limited region $\gamma \in \bigcap[\nabla]_\infty$ and hence a limited region $\alpha \in \bigcap[\nabla]_\infty$ with $\alpha \ll (\beta \wedge \gamma)$. So, by Lemma 4.2(d), $\mathbf{C}(\alpha) \subseteq \mathbf{I}(\beta)$ and consequently $\mathbf{C}(\alpha) \subseteq O$, and by Definition 4.1(a), $[\nabla]_\infty \in \mathbf{I}(\alpha)$. (b) by (a), given that $\mathbf{I}(1)$ is open. □

LEMMA 4.11. *Let* $\alpha \neq 0$ *be limited and* $\mathbf{F}$ *be a proper filter on* $P_{\mathcal{R}}$ *with* $\mathbf{C}(\alpha) \in \mathbf{F}$. *Then there is a point* $[\nabla]_\infty \in \mathbf{C}(\alpha)$ *such that every open set* O *with* $[\nabla]_\infty \in O$ *overlaps with every member of* $\mathbf{F}$.

Proof. Define a set Γ of regions: $\gamma \in \Gamma$ if and only if $\mathbf{I}(\gamma) \in \mathbf{F}$. It is easily established that Γ is a proper filter on Ω. For any $\gamma \in \Gamma$, $\gamma \wedge \alpha \neq 0$ by Lemma 4.2(c), since both $\mathbf{I}(\gamma)$ and $\mathbf{C}(\alpha)$ belong to $\mathbf{F}$. Let Σ' be the filter derived from the filter basis $\{\alpha \wedge \gamma \mid \gamma \in \Gamma\}$. Then Γ' is a proper limited filter with $\alpha \in \Gamma'$. Let ∇ be any ultrafilter with $\Gamma' \subseteq \nabla$. Since $\alpha \in \nabla$, $[\nabla]_\infty \in \mathbf{C}(\alpha)$. Let O be an open subset of $P_{\mathcal{R}}$ with $[\nabla]_\infty \in O$. Suppose $A \in \mathbf{F}$ but $O \cap A = \varnothing$. Then by Lemma 4.10(a) there is a region β such that $[\nabla]_\infty \in \mathbf{I}(\beta)$ and $\mathbf{C}(\beta) \subseteq O$. But then $A \subseteq \mathbf{I}(-\beta)$ by Lemma 4.1, hence $\mathbf{I}(-\beta) \in \mathbf{F}$, hence $-\beta \in \Gamma$, hence $-\beta \in \nabla$, and hence $[\nabla]_\infty \in \mathbf{C}(-\beta)$. But this is impossible by Lemma 4.1. Hence $O \cap A \neq \varnothing$, for every open set O with $[\nabla]_\infty \in O$ and every $A \in \mathbf{F}$. □

LEMMA 4.12. *Let* α *be unlimited. Then there exists a proper filter* $\mathbf{F}$ *on* $P_{\mathcal{R}}$ *with* $\mathbf{C}(\alpha) \in \mathbf{F}$ *such that for every point* $[\nabla]_\infty \in \mathbf{C}(\alpha)$ *there is an open set* O, $[\nabla]_\infty \in O$, *that does not overlap with some member of* $\mathbf{F}$.

Proof. Consider the set $\Gamma = \{\gamma \mid \alpha \wedge -\gamma \text{ is limited}\}$. It is easily verified that, by virtue of constraints A6, A7, and A8, Γ is a proper unlimited filter with α as a member. Let $\mathbf{F}$ be the family of sets $\{A \mid \mathbf{I}(\gamma) \subseteq A$, for some γ in $\Gamma\}$. Then $\mathbf{F}$ is a proper filter on $P_{\mathcal{R}}$ with $\mathbf{I}(\alpha)$, and hence, by Lemma 4.2, $\mathbf{C}(\alpha)$, as elements. Let $[\nabla]_\infty$ be any point of $P_{\mathcal{R}}$. Then there is by Lemma 4.10(b) a limited region β such that $[\nabla]_\infty \in \mathbf{I}(\beta)$. $\mathbf{I}(\beta)$ is an open set of $P_{\mathcal{R}}$, and since β is limited, so is $\alpha \wedge \beta$ by constraint A7, and consequently $\alpha \wedge -\beta \in \Gamma$. Hence $\mathbf{I}(\alpha \wedge -\beta) \in \mathbf{F}$. But $\mathbf{I}(\beta) \cap \mathbf{I}(\alpha \wedge -\beta) = \mathbf{I}(\beta \wedge (\alpha \wedge -\beta)) = \mathbf{I}(0) = \varnothing$ by Lemmas 4.3(a) and 4.4(a). □

We can prove now that limited regions of $\mathcal{R}$ correspond to compact subsets of $P_{\mathcal{R}}$.

THEOREM 4.8. *α is limited if and only if $\mathbf{C}(\alpha)$ is compact.*

Proof. (i) Suppose α is limited. If $\alpha = 0$ then $\mathbf{C}(\alpha) = \varnothing$, which is trivially compact. If $\alpha \neq 0$, $\mathbf{C}(\alpha)$ is compact by Lemmas 4.9 and 4.11. (ii) Suppose $\mathbf{C}(\alpha)$ is compact. Then α must be limited by Lemma 4.12. □

We are now in a position to establish that if $\mathcal{R}$ is a region-based topology then the point-based topology $P_{\mathcal{R}}$ is locally compact and a T_2 space. Definitions of *locally compact* and T_2 *space*, as well as T_3 *space*, follow.

DEFINITION 4.4. Let $\mathcal{P} = \langle P, \mathbf{C} \rangle$ be a point-based topology. Then

(a) $\mathcal{P}$ is *locally compact* if and only if for every point $x \in P$ there exists a compact closed set C such that $x \in \operatorname{In} C$;
(b) $\mathcal{P}$ is a T_2 *space* if and only if for any distinct points x and x' in P there are disjoint open subsets O and O' of P containing x and x', respectively;
(c) $\mathcal{P}$ is a T_3 *space* if and only if for any point x in P and closed subset C of P with $x \notin C$ there are disjoint open subsets O and O' of P such that $x \in O$ and $C \subseteq O'$.

THEOREM 4.9. *$\mathcal{P}_{\mathcal{R}}$ is locally compact.*

Proof. Let $[\nabla]_\infty$ be any point in $P_{\mathcal{R}}$. By Lemma 4.10(b) there is then a limited region α with $[\nabla]_\infty \in \mathbf{I}(\alpha)$, i.e. $[\nabla]_\infty \in \operatorname{In} \mathbf{C}(\alpha)$ by Lemma 4.7(b). $\mathbf{C}(\alpha)$ is compact by Theorem 4.8. So $\mathcal{P}_{\mathcal{R}}$ is locally compact by Definition 4.4. □

THEOREM 4.10. *$\mathcal{P}_{\mathcal{R}}$ is a* T_2 *space.*

Proof. Let $[\nabla_1]_\infty$ and $[\nabla_2]_\infty$ be distinct points of the space $P_{\mathcal{R}}$. By the definition of *point* and Lemma 2.5 there exist then a limited region $\alpha \in \nabla_1$ and a region $\beta \in \nabla_2$ such that $\alpha \not\bigcirc \beta$. Hence $[\nabla_1]_\infty \in \mathbf{C}(\alpha)$ and $[\nabla_2]_\infty \in \mathbf{C}(\beta)$. By Lemma 1.4 there is a region γ such that $\alpha \ll \gamma$ and $\beta \ll -\gamma$. Hence $[\nabla_1]_\infty \in \mathbf{I}(\gamma)$ and $[\nabla_2]_\infty \in \mathbf{I}(-\gamma)$ by Lemma 4.2(d). $\mathbf{I}(\gamma)$ and $\mathbf{I}(-\gamma)$ are open, and $\mathbf{I}(\gamma) \cap \mathbf{I}(-\gamma) = \varnothing$ by Lemmas 4.1 and 4.2(a). □

So, given the definitions of *point* and *closed set* in Section 2 and this section, every region-based topology $\mathcal{R}$ generates a point-based topology $P_{\mathcal{R}}$ that is locally compact and T_2.

5. Generating a Region-Based Topology

Given a region-based topology one can identify a point-based topology, locally compact and T_2, that uniquely corresponds to it (Theorems 4.5, 4.9, and 4.10). Moreover, the proof of the result shows how one can, conversely, obtain the corresponding region-based topology from such a point-based one: take as regions the regular-closed sets and define the mereological operations in accordance with Theorem 4.6, define connection as overlap, and limitedness as compactness. What is not clear yet is whether *every* point-based topology that is locally compact and T_2 corresponds to a region-based topology by virtue of the definitions just noted. Theorem 5.6 below will settle the question.

The proofs of that theorem and others in this section make use of some results from point-set topology which will be listed here, without proof, as Lemma 5.1.[12]

LEMMA 5.1. (a) *If A is open, then* $A \cap \mathrm{Cl}\, B \subseteq \mathrm{Cl}(A \cap B)$ *and* $\mathrm{Cl}(A \cap \mathrm{Cl}\, B) = \mathrm{Cl}(A \cap B)$;
(b) *If O is open, C is closed, and $O \subseteq C$, then* $\mathrm{Cl}\, O$ *is regular-closed and* $\mathrm{In}(C)$ *is regular-open;* $O \subseteq \mathrm{In}\,\mathrm{Cl}\, O \subseteq \mathrm{Cl}\, O \subseteq \mathrm{Cl}\,\mathrm{In}\, C \subseteq C$; *and* $O \subseteq \mathrm{In}\,\mathrm{Cl}\, O \subseteq \mathrm{In}\, C \subseteq \mathrm{Cl}\,\mathrm{In}\, C \subseteq C$;
(c) *If A is compact, B is closed and $B \subseteq A$, then B is compact;*
(d) *If A and B are compact, so is* $A \cup B$;
(e) *A locally compact* T_2 *space is a* T_3 *space;*
(f) *In a locally compact* T_2 *space the regular-closed sets form a basis of the topology;*
(g) *C is compact if and only if when* $\mathbf{G}$ *is a family of closed sets and* $C \cap \bigcap \mathbf{G} = \varnothing$, *then there is a finite subfamily* $\mathbf{G}'$ *of* $\mathbf{G}$ *with* $C \cap \bigcap G' = \varnothing$.

As is well-known, the regular-closed sets of a topological space constitute a Boolean algebra. The next theorem records the details of this algebra.

THEOREM 5.1. *Let* $\mathcal{P} = \langle P, \mathbf{C} \rangle$ *be a point-based topology. Then the regular-closed sets of* $\mathcal{P}$ *constitute a complete Boolean algebra with respect to the elements, the relation, and the operations defined by*

(a) $0 = \varnothing$ *and* $1 = P$
(b) $A \leq B$ *if and only if* $A \subseteq B$
(c) $-A = \mathrm{Cl} \setminus A$;
(d) $A \vee B = A \cup B$;
(e) $A \wedge B = \mathrm{Cl}(\mathrm{In}\, A \cap \mathrm{In}\, B)$;
(f) $\mathbf{V}\mathbf{G} = \mathrm{Cl} \bigcup \{A \mid A \in \mathbf{G}\}$;
(g) $\Lambda\mathbf{G} = \mathrm{Cl}\,\mathrm{In} \bigcap \{A \mid A \in \mathbf{G}\}$.

So, given a point-based topology $\mathcal{P}$ we can define a structure $\mathcal{R}_{\mathcal{P}}$, consisting of a Boolean algebra $\Omega_{\mathcal{P}}$, with a connection relation $\infty_{\mathcal{P}}$ and a class $\Delta_{\mathcal{P}}$ of limited elements. The definition is guided by Theorems 4.7 and 4.8.

DEFINITION 5.1. Let $\mathcal{P} = \langle P, \mathbf{C} \rangle$ be a point-based topology. Then $\mathcal{R}_{\mathcal{P}}$ is the structure $\langle \Omega_{\mathcal{P}}, \infty_{\mathcal{P}}, \Delta_{\mathcal{P}} \rangle$, where $\Omega_{\mathcal{P}}$ is the Boolean algebra of regular-closed sets of $\mathcal{P}$, $A \infty_{\mathcal{P}} B$ if and only if $A \cap B \neq \varnothing$; and $A \in \Delta_{\mathcal{P}}$ if and only if A is compact.

If $\mathcal{P}$ and $\mathcal{P}'$ are two point-based topologies that are homeomorphic, then the corresponding structures $\mathcal{R}_{\mathcal{P}}$ and $\mathcal{R}_{\mathcal{P}'}$ are obviously isomorphic. That $\mathcal{P}$ and $\mathcal{P}'$ are homeomorphic when $\mathcal{R}_{\mathcal{P}}$ and $\mathcal{R}_{\mathcal{P}'}$ are isomorphic cannot be proved without further assumptions. Before reaching the result in Theorem 5.8 we need to establish that $\mathcal{R}_{\mathcal{P}}$ constitutes a region-based topology if $\mathcal{P}$ is locally compact and T_2.

THEOREM 5.2. *Let $\mathcal{P}$ be any point-based topology. Then $\mathcal{R}_{\mathcal{P}}$ meets constraints* A1–A5.

Proof. With $A \infty B$ defined as $A \cap B \neq \varnothing$ A1–A5 are obvious truths of set theory. □

THEOREM 5.3. *Let $\mathcal{P}$ be any point-based topology. Then $\mathcal{R}_{\mathcal{P}}$ meets constraints* A6–A8.

Proof. (a) A6 is trivial, since 0, being the empty set, is compact. (b) Given that *A is limited* is defined as *A is compact*, A7 follows by Lemma 5.1(c). (c) A8 follows from Lemma 5.1(d). □

While $\mathcal{R}_{\mathcal{P}}$ meets constraints A1–A8, whatever the point-based topology $\mathcal{P}$, $\mathcal{R}_{\mathcal{P}}$ can be shown to meet A9 only if $\mathcal{P}$ is assumed to be locally compact.

LEMMA 5.2. *If $\mathcal{P} = \langle P, \mathbf{C} \rangle$ is locally compact and $x \in P$ then there exists a regular-closed, compact set D such that $x \in \operatorname{In} D$.*

Proof by Definition 4.4 and Lemma 5.1(b).

LEMMA 5.3. *Let $\mathcal{P} = \langle P, \mathbf{C} \rangle$ be locally compact, A and B regular-closed subsets of P, and $A \cap B \neq \varnothing$. Then there is a compact, regular-closed subset B' of B such that $A \cap B' \neq \varnothing$.*

Proof. Let x be a point belonging to both A and B. Then there exists, by Lemma 5.2, a compact regular-closed set D such that $x \in \operatorname{In} D$. Since B is regular-closed, $x \in \operatorname{In} D \cap \operatorname{Cl} \operatorname{In} B$. Hence $x \in B' = \operatorname{Cl}(\operatorname{In} D \cap \operatorname{In} B) = \operatorname{Cl} \operatorname{In}(D \cap B)$ by Lemma 5.1(a). Hence $A \cap B' \neq \varnothing$. B' is regular-closed by Theorem 5.1(e), $B' \subseteq B$, and $B' \subseteq D$ by Lemma 5.1(b). So, B' is compact by Lemma 5.1(c). □

THEOREM 5.4. *Let $\mathcal{P}$ be locally compact. Then $\mathcal{R}_{\mathcal{P}}$ meets constraint* A9.

Proof. Given the definitions of $A \infty B$ as $A \cap B \neq \varnothing$ and of *A is limited* as *A is compact*, $\mathcal{R}_\mathcal{P}$ meets constraint A9 by Lemma 5.3 and Theorem 5.1(b). □

As to A10, $\mathcal{R}_\mathcal{P}$ can be shown to meet this constraint only if $\mathcal{P}$ is assumed to be both locally compact and T_2.

LEMMA 5.4. *Let $\mathcal{P} = \langle P, \mathbf{C} \rangle$ be a locally compact* T_2 *space; let x be in O with O an open subset of P. Then there exists a regular-closed and compact set C such that $x \in \text{In}\, C$ and $C \subseteq O$.*

Proof. $\backslash O$ is closed and $x \notin \backslash O$. Since $\mathcal{P}$ is a T_3 space by Lemma 5.1(e), there are open sets O_1 and O_2 such that $x \in O_1$, $\backslash O \subseteq O_2$ and $O_1 \cap O_2 = \varnothing$. By Lemma 5.1(b) $\text{Cl}\, O_1$ is regular-closed. So, since $\text{Cl}\, O_1 \subseteq \backslash O_2$, $\text{Cl}\, O_1 \subseteq O$, and so $x \in \text{In}\,\text{Cl}\, O_1 \subseteq \text{Cl}\, O_1 \subseteq O$. By Lemma 5.2 there exists a regular-closed, compact set D such that $x \in \text{In}\, D$. $C = \text{Cl}(\text{In}\, D \cap \text{In}\,\text{Cl}\, O_1)$ is regular-closed by Theorem 5.1(e) and compact by Lemma 5.1(c). And $x \in \text{In}\, C$ and $C \subseteq O$. □

LEMMA 5.5. *Let $\mathcal{P} = \langle P, \mathbf{C} \rangle$ be a locally compact* T_2 *space, A and B regular-closed subsets of P with $A \subseteq \text{In}\, B$, A compact, and $B \neq \varnothing$. Then there exists a regular-closed and compact set $C \neq \varnothing$ such that $A \subseteq \text{In}\, C$ and $C \subseteq \text{In}\, B$.*

Proof. (i) Suppose $A = \varnothing$. Since B is regular-closed and non-empty, $\text{In}\, B \neq \varnothing$. Taking x to be any point in $\text{In}\, B$, it follows by Lemma 5.4 that there is a non-empty, regular-closed and compact set C such that $\varnothing \subseteq \text{In}\, C$ and $C \subseteq \text{In}\, B$. (ii) Suppose $A \neq \varnothing$. For every $x \in A$ there exists by Lemma 5.4 a regular-closed, compact set C_x such that $x \in \text{In}\, C_x$ and $C_x \subseteq \text{In}\, B$. Since A is compact, finitely many of the sets $\text{In}\, C_x$ cover A. Let C be the union of their closures. Then C is non-empty, is regular-closed by Theorem 5.1(d), and compact by Lemma 5.1(d). Moreover, $A \subseteq \text{In}\, C$ and $C \subseteq \text{In}\, B$. □

THEOREM 5.5. *Let $\mathcal{P}$ be a locally compact* T_2 *space. Then $\mathcal{R}_\mathcal{P}$ meets constraint* A10.

Proof. Given the definition of *A is limited* as *A is compact* and of $A \infty B$ as $A \cap B \neq \varnothing$, hence of $A \ll B$ as $A \subseteq \text{In}\, B$, $\mathcal{R}_\mathcal{P}$ meets constraint A10 by Lemma 5.5. □

Theorems 5.2, 5.3, 5.4, and 5.5 together amount to

THEOREM 5.6. *If $\mathcal{P}$ is a point-based topology that is locally compact and satisfies* T_2 *then $\mathcal{R}_\mathcal{P}$ is a region-based topology.*

In order to complete the proof of the one-to-one correspondence (modulo isomorphism and homeomorphism) between region-based topologies and locally compact T_2 spaces it needs to be shown that the construction that generates a point-based topology from a region-based one is the inverse of that which generates a region-based topology from a locally compact T_2 space, or, succinctly, that $\mathcal{P} = \langle P, \mathbf{C} \rangle$ is homeomorphic to $\mathcal{P}_{\mathcal{R}_{\mathcal{P}}} = \langle P_{\mathcal{R}_{\mathcal{P}}}, \mathbf{C}_{\mathcal{R}_{\mathcal{P}}} \rangle$. And to that end we need to establish first that the points of $\mathcal{P}$ match one-to-one the points as defined within $\mathcal{R}_{\mathcal{P}}$. For the latter we choose the characterisation of points in terms of maximal proper, limited and contracting filters introduced in Section 3. And we show that for any point x of $\mathcal{P}$ the family of sets $\mathbf{F}(x) = \{A \mid A$ is regular-closed and $x \in \text{In}\, A\}$ is such a filter, that any such filter is $\mathbf{F}(x)$ for some x, and that $x = y$ when $\mathbf{F}(x) = \mathbf{F}(y)$.

DEFINITION 5.2. Let $\mathcal{P} = \langle P, \mathbf{C} \rangle$ be a point-based topology and $x \in P$. Then $\mathbf{F}(x)$ is the family $\{A \mid A$ is regular-closed and $x \in \text{In}\, A\}$ of subsets of P.

LEMMA 5.6. *Let $\mathcal{P} = \langle P, \mathbf{C} \rangle$ be a locally compact T_2 space and x a point of P. Then*

(a) *If $\mathbf{F}(x) = \mathbf{F}(y)$, then $x = y$;*
(b) *$\mathbf{F}(x)$ is a maximal proper, limited and contracting filter on the Boolean algebra $\Omega_{\mathcal{P}}$ in the region-based topology $\mathcal{R}_{\mathcal{P}}$.*

Proof. (a) Suppose $x \neq y$. Since $\mathcal{P}$ is a T_2 space, there are open subsets O_1 and O_2 of P such that $x \in O_1$, $y \in O_2$, and $O_1 \cap O_2 = \varnothing$. $x \in \text{In}\,\text{Cl}\, O_1$ by Lemma 5.1(b). Hence $\text{Cl}\, O_1 \in \mathbf{F}(x)$. But $\text{Cl}\, O_1 \notin \mathbf{F}(y)$ since $y \notin \text{Cl}\, O_1$. Hence $\mathbf{F}(x) \neq \mathbf{F}(y)$. (b) $\mathbf{F}(x)$ is easily seen to be a filter on $\Omega_{\mathcal{P}}$; it is proper since $\text{In}\,\varnothing = \varnothing$, and limited and contracting by Lemma 5.4. Suppose $\mathbf{F}(x) \subset \mathbf{G}$, where $\mathbf{G}$ is a contracting filter on $\Omega_{\mathcal{P}}$. Then there exists a regular-closed subset B of P such that $B \in \mathbf{G}$ but $B \notin \mathbf{F}(x)$. Then $x \notin \text{In}\, B$, i.e. $x \in \text{Cl}\,\backslash B$. Since $\mathbf{G}$ is contracting, there is a regular-closed set $B' \in \mathbf{G}$ with $B' \subseteq \text{In}\, B$. Hence $x \in \backslash B'$; so $\text{Cl}\,\backslash B' \in \mathbf{F}(x)$ and $\text{Cl}\,\backslash B' \in \mathbf{G}$. Hence $B' \wedge \text{Cl}\,\backslash B' \in \mathbf{G}$. Now $B' \wedge \text{Cl}\,\backslash B' = \text{Cl}(\text{In}\, B' \cap \backslash B') = \varnothing$. Hence, by the definition of the null element 0 of $\Omega_{\mathcal{P}}$ as $\varnothing$, $\mathbf{G}$ is not a proper filter on $\Omega_{\mathcal{P}}$. So $\mathbf{F}(x)$ is a maximal proper, limited and contracting filter on $\Omega_{\mathcal{P}}$. □

LEMMA 5.7. *Let $\mathcal{P}$ be a locally compact* T_2 *space. If* $\mathbf{G}$ *is a maximal proper, limited and contracting filter on $\Omega_{\mathcal{P}}$ then there is exactly one point x of $\mathcal{P}$ such that $\mathbf{G} = \mathbf{F}(x)$.*

Proof. Suppose that $\mathbf{G}$ is a maximal proper, limited and contracting filter on $\Omega_{\mathcal{P}}$ and that $C \in \mathbf{G}$ is compact. Assume $\bigcap \mathbf{G} = \varnothing$. Since $C \in \mathbf{G}$ there is a finite subset $\mathbf{G}'$ of $\mathbf{G}$ such that $\bigcap \mathbf{G}' = \varnothing$, this by Lemma 5.1(g). So $\mathrm{Cl}\,\mathrm{In} \bigcap \mathbf{G}' = \varnothing$, i.e., by Theorem 5.1(g), $\Lambda \mathbf{G}' = 0$. But this is impossible since $\mathbf{G}$ is a proper filter on $\Omega_{\mathcal{P}}$. Hence $\bigcap \mathbf{G} \neq \varnothing$. Suppose $x \in \bigcap \mathbf{G}$ and $A \in \mathbf{G}$. Then there exists in $\mathbf{G}$ a regular-closed set $B \subseteq \mathrm{In}\, A$, since $\mathbf{G}$ is a contracting filter on $\Omega_{\mathcal{P}}$. So $x \in B$, and therefore $x \in \mathrm{In}\, A$. Hence $A \in \mathbf{F}(x)$. Consequently, $\mathbf{G} \subseteq \mathbf{F}(x)$. Hence by Lemma 5.6(b), given that $\mathbf{G}$ is assumed to be maximal, $\mathbf{G} = \mathbf{F}(x)$. □

The totality of maximal proper, limited and contracting filters on $\Omega_{\mathcal{P}}$ constitutes the space $P_{\mathcal{R}_{\mathcal{P}}}$ of points of the point-based topology $\mathcal{P}_{\mathcal{R}_{\mathcal{P}}}$. By Lemmas 5.6 and 5.7 $P_{\mathcal{R}_{\mathcal{P}}} = \{\mathbf{F}(x) \mid x \in P\}$ and $\mathbf{F}$ establishes a one-to-one correlation between P and $P_{\mathcal{R}_{\mathcal{P}}}$. It remains to be shown that under this correlation the closed sets of $\mathcal{P}$ correspond to the closed sets of $\mathcal{P}_{\mathcal{R}_{\mathcal{P}}}$. But since the regular-closed sets form a basis of the topology, it suffices to show that the regular-closed sets of $\mathcal{P}$ correspond to the regular-closed sets of $\mathcal{P}_{\mathcal{R}_{\mathcal{P}}}$, i.e. that, for any subset A of P, $\mathbf{F}(A) = \{\mathbf{F}(x) \mid x \in A\}$ is regular-closed if and only if A is. By Theorem 4.4 the regular-closed sets of $\mathcal{P}_{\mathcal{R}_{\mathcal{P}}}$ are the sets $\mathbf{C}(\alpha)$, α a region in $\Omega_{\mathcal{P}}$. But by Definition 5.1 the regions in $\Omega_{\mathcal{P}}$ are just the regular-closed sets of $\mathcal{P}$. So let A be a regular-closed subset of P. Then, by virtue of the comments on Definition 4.1, $\mathbf{C}(A) = \{\mathbf{F}(x) \mid \mathrm{Cl} \setminus A \notin \mathbf{F}(x)\}$. But given the definition of $\mathbf{F}(x)$, $\mathrm{Cl} \setminus A \notin \mathbf{F}(x)$ just when $x \in A$. Hence $\mathbf{C}(A) = \mathbf{F}(A)$. So $\mathbf{F}$ indeed maps the regular-closed subsets of P onto the regular-closed subsets of $P_{\mathcal{R}_{\mathcal{P}}}$. Hence $\mathcal{P}$ and $\mathcal{P}_{\mathcal{R}_{\mathcal{P}}}$ are homeomorphic.

THEOREM 5.7. *Let $\mathcal{P}$ be a locally compact* T_2 *space. Then $\mathcal{P}$ is homeomorphic to $\mathcal{P}_{\mathcal{R}_{\mathcal{P}}}$.*

So, given Theorem 4.5,

THEOREM 5.8. *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be locally compact* T_2 *spaces. Then $\mathcal{P}_1$ is homeomorphic to $\mathcal{P}_2$ if and only if $\mathcal{R}_{\mathcal{P}_1}$ and $\mathcal{R}_{\mathcal{P}_2}$ are isomorphic,*

and

THEOREM 5.9. *Let $\mathcal{R}$ be a complete region-based topology. Then $\mathcal{R}$ is isomorphic to $\mathcal{R}_{\mathcal{P}_{\mathcal{R}}}$.*

The results of this section are summed up in the

MAIN THEOREM. *The complete region-based topologies correspond one-to-one (modulo isomorphisms and homeomorphisms, respectively) to the locally compact* T_2 *spaces.*

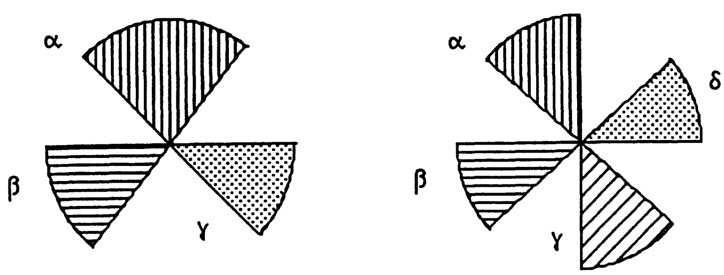
In the light of this result it seems appropriate to ignore from now on region-based topologies that are based on incomplete Boolean algebras. So, in the remaining sections all region-based topologies considered will be complete ones.

The results in the previous sections have shown that there is a close correspondence between region-based topology and point-based topology. If the Boolean algebra underlying region-based topology is assumed to be complete, then Theorems 5.8 and 5.9 state that either is obtainable from the other, modulo isomorphisms and homeomorphisms, respectively. This means that one and the same structure is describable in terms of the parts of the space and in terms of the points located in it. While the existence or non-existence of certain infinite joins or meets of parts must be acknowledged as an important feature of the space, and one that cannot be represented in point-based topology, it also seems appropriate to note that it does hardly affect the structure of the space, the ways in which its parts hang together, i.e. those characteristics that are usually regarded as topological. Hence, assuming completeness, we have an exact correspondence between region-based topologies as defined in Section 1 and point-based topologies that are locally compact and T_2.

What needs to be mentioned, of course, is that neither the point-based topological characterisations nor, apparently, the region-based ones mentioned in those theorems are of the most general kind. Not all topological spaces are locally compact T_2 spaces. A point-based topology that is not both locally compact and T_2 has no region-based counterpart, or no unique (modulo isomorphism) region-based counterpart. Its structure cannot be captured by a description in terms of the connection or absence of connection of parts of the space, where this relation is governed by constraints A1–A10.

It is clear that among the point-based topologies that are not both locally compact and T_2 there are some that cannot be interpreted as descriptions of conceivable spaces. So, if P is $\{a, b, c, d\}$ and the closed sets are $\varnothing$, $\{a, b\}$, $\{c, d\}$, and P, this topology fails to meet T_2. The spatial interpretation of the topology, if there is one at all, will be the same as for the topology on $P' = \{a, c\}$ whose closed sets are $\varnothing$, $\{a\}$, $\{c\}$, and P', namely a space consisting of two atomic, disconnected regions. The duplication of points is a feature of the first topology that has no intuitive spatial significance and so is not representable in region-based topology. On the other hand, the set **Q** of rational numbers with the usual topology, is not locally compact. The regular-closed subsets of **Q** stand in one-to-one correlation with the regular-closed subsets of **R**. So, the spatial interpretation of **Q** is indistinguishable from that of **R**, and the fact that all irrational points are missing from **Q** is not conceivable as a feature of space. Structures like P and **Q** are only mathematical models of a theory (point-set topology) which has great appeal because of its simplicity and elegance but which, without the addition of more specific constraints, viz. local compactness and T_2, has models that cannot be interpreted as being as wholes descriptions of spatial arrangements.

The example shows that without local compactness and T_2 different point-based topologies can give rise to the same region-based topology. Analogously, it is not difficult to see that without A10 it is not guaranteed that distinct point-based topologies correspond to distinct region-based topologies. A particularly simple example is this.



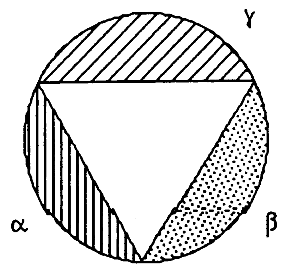
I   II

I and II are meant as representations of the whole space, so that, in the first case, $1 = \alpha \vee \beta \vee \gamma$. The parts (i.e. α, β, and γ in the first case, α, β, γ, and δ in the second) are indivisible or atomic parts of the respective space. Theorem 6.3 in the next section implies that if the space is coherent as a whole the presence of atomic parts amounts to the falsity of axiom A10. With these atomic parts declared limited, axioms A1–A9 are clearly satisfied. Since in both cases the ultrafilters are just those filters that are generated by the atoms, and these ultrafilters are all mutually connected, each structure gives rise to a 1-point space, as is also obvious from the diagrams themselves.

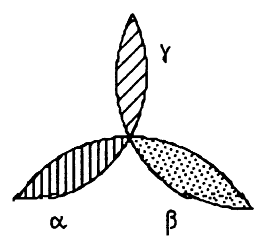
So it is quite clear that from the point-topological description the number of atomic regions cannot be inferred. The description provided with the means of region-based topology is fuller than that provided by point-based topology. So, in the absence of A10 distinct region-based topologies give rise to the same point-based topology. The inclusion of this axiom is then justified by the foregoing considerations. Its presence guarantees an exact correspondence between region-based and point-based descriptions.

Now the fact of this correspondence by itself is no reason not to investigate weaker systems of region-based topology, systems without A10. Indeed, the promise of being able to describe in terms of regions spatial features that point-based topology is not capable of giving an unambiguous account of is one of the motivations for pursuing region-based topology. And there are descriptions of that kind. The last example shows that we can make sense of topological structures in which A10 fails; extendedness without divisibility is a conceptual possibility.

However, it emerges that the approach that has been employed so far, which bases topological descriptions on a 2-place relation of connection is itself not capable of characterising without ambiguity the more general structures that allow for atomic but extended regions of space. The point is illustrated by the following two diagrams.



I



II

In *both* cases $\alpha \infty \beta$, $\beta \infty \gamma$, and $\gamma \infty \alpha$ all hold; but the structures are evidently different. There is obviously no way of distinguishing them from one another by means of the 2-place relation of connection.

In order to enable us to deal with such cases a different approach must be taken. The 2-place relation needs to be replaced by a many-place relation that is immune to permutations, i.e. a property of sets of regions. So, $\infty \Gamma$ is to have the significance that the members of Γ are connected (and not just mutually connected); i.e. there is a point that is coincident with each of them. The constraints that this polyadic relation is subject to are briefly discussed in the Appendix.[13] It is also shown there that if the dyadic relation ∞ satisfies axiom A10 then polyadic connection is definable in terms of dyadic connection.

So, the system of axioms of Section 1 has a claim to the status of preferred system of region-based topology in that it is just strong enough to allow the description of topological structures by means of a dyadic relation of connection. As has already been noted, all topological structures that are models of this system can equally well be described in point set theoretical terms. However, when it comes to topological mappings, region-based topology proves to be more powerful in that there are functions that can be described within region-based topology but not in terms of functions that take points as arguments. These results will be presented and discussed in Section 8.

6. Continuity, Infinite Divisibility and Convexity

In this section I consider some features that are associated with the continuity of a space. These are characteristics that are not included in the minimal characterisation of region-based topology by A1–A10. The notion of continuity of a space is not altogether clear. I suggest that what underlies the concept of continuity is the idea that in a continuous space smooth motion is possible, motion that does not require jumps.

A first requirement would seem to be that $\mathcal{R}$ is complete. I have already indicated that all region-based topologies to be considered in this and subsequent sections are assumed to be complete. So, in what follows I shall mean by a region-based topology $\mathcal{R}$ any structure $\langle \Omega, \infty, \Delta \rangle$, with Ω a complete Boolean algebra, that meets A1–A10.

Another feature that must play a role in connection with continuity is infinite divisibility, since indivisible, i.e. smallest (or atomic), parts of space would be obstacles to smooth motion. Infinite divisibility is not guaranteed by the axioms A1–A10. Let Ω be any *atomic* Boolean algebra, let $\alpha \infty \beta$ hold only when $\alpha \wedge \beta \neq 0$, and let Δ contain every element of Ω. Them $\mathcal{R} = \langle \Omega, \infty, \Delta \rangle$ meets all constraints A1–A10. Any region-based topology of this type corresponds to a totally disconnected space $\mathcal{P}_{\mathcal{R}}$. Since then infinite divisibility is not implied by the basis set of axioms we make it the content of a new axiom:

B1 *If* $\alpha \neq 0$ *then there is a region* $\beta \neq 0$ *with* $\beta < \alpha$.

It can be shown now that if a region-based topology is infinitely divisible then no singleton set of its point-based counterpart is open.

THEOREM 6.1. *Let* $\mathcal{R}$ *be a region-based topology that meets* B1. *Then no* 1*-point set of* $\mathcal{P}_{\mathcal{R}}$ *is open.*

Proof. Let $[\nabla]_\infty$ be a point in $P_{\mathcal{R}}$ and suppose that $\{[\nabla]_\infty\}$ is open. Then $\{[\nabla]_\infty\} = \mathsf{I}(\alpha)$ for some region $\alpha \neq 0$, this by Lemmas 4.6(b) and 4.4(a). By B1 there exists then a region $\beta \neq 0$ such that $\beta < \alpha$. And by Lemma 4.4(c), Lemma 4.2(b), and Theorem 4.1(a) $\mathsf{I}(\beta)$ must be both different from the empty set and properly included in $\{[\nabla]_\infty\}$, which is impossible. So $\{[\nabla]_\infty\}$ is not open. □

Consequently, when a region-based topology $\mathcal{R}$ satisfies B1, no finite set of points of $\mathcal{P}_\mathcal{R}$ is open, and hence every regular-closed set of points of $\mathcal{P}_\mathcal{R}$, other than $\varnothing$, has infinitely many members. Conversely, if $\mathcal{P}_\mathcal{R}$ has no open 1-point sets, then $\mathcal{R}$ meets B1.

LEMMA 6.1. *Let $\mathcal{P} = \langle P, \mathbf{C} \rangle$ be a* T_2 *space in which no singleton $\{x\}$ is open. Then if $A \neq \varnothing$ is regular-closed set in P, there is a regular-closed set $B \neq \varnothing$ such that $B \subset A$.*

Proof. Suppose $A \neq \varnothing$ is regular-closed. Then $\mathrm{In}\, A \neq \varnothing$, since $A = \mathrm{Cl}\,\mathrm{In}\, A$. $\mathrm{In}\, A$ is not a singleton set by assumption. So $\mathrm{In}\, A$ contains at least two distinct points x and y. Since $\mathcal{P}$ is a T_2 space there are then open subsets O_1 and O_2 of $\mathrm{In}\, A$ such that $O_1 \cap O_2 = \varnothing$, $x \in O_1$, and $y \in O_2$. $B = \mathrm{Cl}\, O_1$ is regular-closed by Lemma 5.1(b) and $y \notin B$. Hence $B \subset A$. □

By virtue of Theorems 4.10, 4.4 and 4.6(a) this lemma implies

THEOREM 6.2. *$\mathcal{R}$ meets* B1 *if no 1-point set of $\mathcal{P}_\mathcal{R}$ is open.*

As we have seen, constraint A9 does not guarantee the infinite divisibility of the topology by itself. It does so, however, in conjunction with the postulate that the space of $\mathcal{R}$, i.e. the element 1 of Ω, be *coherent*, provided that Ω is not the 2-element Boolean algebra $\{1, 0\}$, i.e. provided that the space of $\mathcal{R}$ is divisible. Note indeed that constraints A1–A10 are all satisfied by this algebra as long as 1 limited. So, we formulate the following two constraints.

B2 *There exists in Ω at least one region α other than* 0 *and* 1.
B3 1 *is coherent.*

THEOREM 6.3. *Let $\mathcal{R}$ meet constraints* B2 *and* B3. *Then $\mathcal{R}$ meets* B1.

Proof. Suppose $\alpha \neq 0$. If $\alpha = 1$, then B1 by B2. So, assume $\alpha \neq 1$. By Lemma 1.3 there is a region $\beta \neq 0$ such that $\beta \leq \alpha$ and $\beta \ll \alpha$. But by B3 and Lemma 1.8 $\beta \neq \alpha$. Hence $\beta < \alpha$. □

So, if the space of the topology $\mathcal{R}$ is coherent then, thanks to the interpolation constraint A9, it is infinitely divisible, if divisible at all.

It is easy to see that when $\mathcal{R}$ meets B2 this amounts to $\mathcal{P}_\mathcal{R}$ having more than one point, while meeting B3 corresponds to $\mathcal{P}_\mathcal{R}$ being *connected.*

DEFINITION 6.1. Let $\mathcal{P} = \langle P, \mathbf{C} \rangle$ be a point-based topology. Then $\mathcal{P}$ is *connected* if there are no closed subsets A and B of P such that $A \cup B = P$, $A \cap B = \varnothing$, and both $A \neq \varnothing$ and $B \neq \varnothing$.

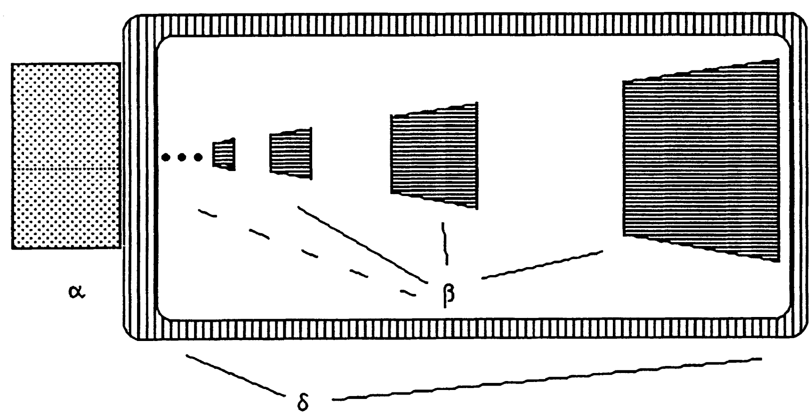
THEOREM 6.4.

(a) $\mathcal{R}$ *meets* B2 *if and only if* $\mathcal{P}_{\mathcal{R}}$ *has more than one point;*
(b) $\mathcal{R}$ *meets* B3 *if and only if* $\mathcal{P}_{\mathcal{R}}$ *is connected.*

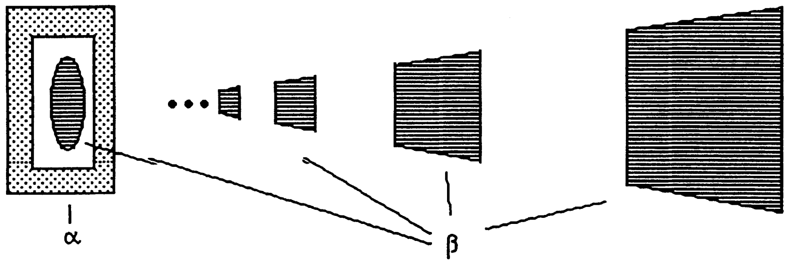
Proof. (a) (i) Suppose there is a region α in Ω other than 0 and 1. Then $-\alpha \neq 0$; and $\mathsf{I}(\alpha) \neq \varnothing$, $\mathsf{I}(-\alpha) \neq \varnothing$ and $\mathsf{I}(\alpha) \cap \mathsf{I}(-\alpha) = \varnothing$ by Lemmas 4.3(a) and 4.4. Hence $P_{\mathcal{R}}$ has at least two points. (ii) Suppose $\mathcal{R} = \langle \Omega, \infty, \Delta \rangle$ fails to meet B2. Then Ω consists of just 0 and 1, and $\{1\}$ is the only ultrafilter on Ω. $\{1\}$ is limited by Lemma 1.3(b). Hence $P_{\mathcal{R}}$ consists of the single point $\{\{1\}\}$. (b) By virtue of Lemmas 4.4 and 4.3(b), and Theorem 4.7(a) 1 is not coherent if and only if $P_{\mathcal{R}}$ is not connected. □

A further type of situation conflicting with smooth motion obtains when α and β are infinitesimally close, as for example in the diagram on page 3, but $\alpha \infty \beta$ does not hold so that, given the definition of points in Section 3, α and β do not even have a boundary point in common. I have indicated before that A1–A10 do not rule out this possibility.[14] What such a situation suggests about motion from β to α is that one could be infinitesimally close to α without having reached α, that nowhere moving to the boundary of β means reaching the boundary of α. So, an interpretation like this would go against the continuity of the space.

In order to formulate a condition that would rule out such an interpretation, a more precise explication in needed of what it is for a region to be infinitesimally close to another. A first thought is that β is infinitesimally close to α if β cannot be separated from α by a convex region δ, i.e. if a situation like that in the following diagram does not obtain.



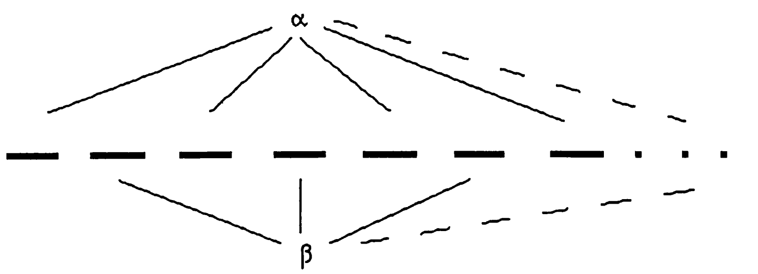
Obviously equivalent to this is the condition that β *cannot* be embedded in a convex region γ which is itself disconnected from α. This suggestion will not do, however, as the following diagram illustrates.



Even though β is *not* infinitesimally close to α there is no convex region γ which both includes β and is disconnected from α. But while embedding of β in a single convex region (that is disconnected from α) is not possible, embedding in a finite number of such regions is possible. So, the following account of 'β is infinitesimally close to α' suggests itself.

> There is no region γ which is the join of finitely many convex regions $\gamma_1, \ldots, \gamma_n$ such that $\beta \leq \gamma$ and $\gamma \not\circ \alpha$.

But this explication is reasonable only when the region β is limited. When β is unlimited, the explicans may be true of β in relation to α even though β is not infinitesimally close to α. Consider the disconnected unlimited regions α and β in this infinite 1-dimensional space:



Any finite join of convex regions that covers the infinitely many convex regions that make up β must also cover parts of α. But α and β are not infinitesimally close. So the proposed definition must be amended by adding the requirement that β be limited.

The account of 'infinitesimally close' permits now formulation of the constraint entertained above, namely that β is connected with α when infinitesimally close to α.

B4 *If β is limited and $\alpha \not\circ \beta$ then there is a region γ, which is the join $\gamma_1 \vee \cdots \vee \gamma_n$ of finitely many convex regions $\gamma_1, \ldots, \gamma_n$, such that $\beta \leq \gamma$ and $\alpha \not\circ \gamma$. (If β is limited and $\beta \ll \alpha$ then there is a finite join $\gamma = \gamma_1 \vee \cdots \vee \gamma_n$ of convex regions $\gamma_1, \ldots, \gamma_n$ such that $\beta \leq \gamma$ & $\gamma \ll \alpha$).*

B4 can be given this more convenient form.

> If β is limited and $\alpha \not\infty \beta$ then there is a region γ, which is the join $\gamma_1 \vee \cdots \vee \gamma_n$ of finitely many convex regions $\gamma_1, \ldots, \gamma_n$, where *the join of any two of these is not convex*, such that $\beta \leq \gamma$ and $\alpha \not\infty \gamma$.

Assuming that this constraint captures in essence what is intended by ‘continuity’, I adopt the following definition.

DEFINITION 6.2. A region-based topology $\mathcal{R}$ is called *continuous* if it meets B4.

Next will be studied some consequences of the definitions (in Section 1) of coherence and convexity and of constraint B4. Not surprisingly, the concept of *coherence* within region-based topology corresponds to the notion of *connectedness* within point-based topology as applied to subsets of the space. *Convexity* also corresponds to *connectedness*, though in a somewhat different way. A set A of points counts as *connected* when A, considered as a subspace of the topology, is connected in accordance with Definition 6.1. And this amounts to the following definition:

DEFINITION 6.3. A set A in a point-based topology is *connected* if and only if there are no closed sets C_1, C_2 such that $C_1 \cap C_2 \cap A = \varnothing$, $C_1 \cap A \neq \varnothing$, $C_2 \cap A \neq \varnothing$, and $A \subseteq C_1 \cup C_2$.

Of course, this definition yields Definition 6.1 when A is the whole space P.

LEMMA 6.2. (a) *Let* $[\nabla]_\infty$ *be a point with* $[\nabla]_\infty \in \mathbf{I}(\alpha) \cap \mathbf{C}(\sigma) \cap \mathbf{C}(\rho)$. *Then there is a region* $\gamma \ll \alpha$ *such that* $(\gamma \wedge \sigma) \infty (\gamma \wedge \rho)$;
(b) *If* $\alpha_1, \ldots, \alpha_n$ *are all convex, but* $\alpha_i \vee \alpha_j$ *is not, unless* $i = j$, *then* $\mathbf{I}(\alpha_1 \vee \cdots \vee \alpha_n) = \mathbf{I}(\alpha_1) \cup \cdots \cup \mathbf{I}(\alpha_n)$.

Proof. (a) By Lemma 4.10(a) and Theorem 4.7(b) $[\nabla]_\infty \in \mathbf{I}(\gamma)$ and $\gamma \ll \alpha$ for some region γ. Since $[\nabla]_\infty \in \mathbf{I}(\gamma) \cap \mathbf{C}(\sigma)$, $[\nabla]_\infty \in \mathbf{C}(\gamma \wedge \sigma)$ by Lemmas 5.1(a) and 4.3(a). Similarly, $[\nabla]_\infty \in \mathbf{C}(\gamma \wedge \rho)$. Hence $(\gamma \wedge \sigma) \infty (\gamma \wedge \rho)$ by Theorem 4.7(a). (b) Suppose $\alpha_1, \ldots, \alpha_n$ are all convex and $\mathbf{I}(\alpha_1 \vee \cdots \vee \alpha_n) \neq \mathbf{I}(\alpha_1) \cup \cdots \cup \mathbf{I}(\alpha_n)$. Obviously $\mathbf{I}(\alpha_1 \vee \cdots \vee \alpha_n) \supseteq \mathbf{I}(\alpha_1) \cup \cdots \cup \mathbf{I}(\alpha_n)$. So, assume that there is a point $[\nabla]_\infty$ with $[\nabla]_\infty \in \mathbf{I}(\alpha_1 \vee \cdots \vee \alpha_n)$, but $[\nabla]_\infty \notin \mathbf{I}(\alpha_i)$ for $i = 1, \ldots, n$. Then there must be at least two regions α_i and α_j $(i \neq j)$ so that $[\nabla]_\infty \in \mathbf{C}(\alpha_i)$ and $[\nabla]_\infty \in \mathbf{C}(\alpha_j)$, since $\alpha_1 \vee \cdots \vee \alpha_n$ belongs to an ultrafilter if and only if at least one of $\alpha_1, \ldots \alpha_n$ does. Now suppose $\sigma \vee \rho = \alpha_i \vee \alpha_j$. If both

$\sigma \wedge \alpha_i \neq 0$ and $\rho \wedge \alpha_i \neq 0$ or both $\sigma \wedge \alpha_j \neq 0$ and $\rho \wedge \alpha_j \neq 0$, then there is a region $\gamma \ll \alpha_i \vee \alpha_j$ such that $(\gamma \wedge \sigma) \infty (\gamma \wedge \rho)$, since α_i and α_j are convex. On the other hand, if $\sigma = \alpha_i$ and $\rho = \alpha_j$ or, equivalently, $\sigma = \alpha_j$ and $\rho = \alpha_i$, then by (a) there is a region $\gamma \ll \alpha_i \vee \alpha_j$ such that $(\gamma \wedge \sigma) \infty (\gamma \wedge \rho)$. So, $\alpha_i \vee \alpha_j$ is convex. □

LEMMA 6.3. *α is coherent if and only if $\mathbf{C}(\alpha)$ is connected.*

Proof. (a) Suppose that $\mathbf{C}(\alpha)$ is not connected. Then there are closed sets C_1 and C_2 such that (i) $C_1 \cap C_2 \cap \mathbf{C}(\alpha) = \varnothing$, (ii) $C_1 \cap \mathbf{C}(\alpha) \neq \varnothing$, (iii) $C_2 \cap \mathbf{C}(\alpha) \neq \varnothing$, and (iv) $\mathbf{C}(\alpha) \subseteq C_1 \cup C_2$. $C_1 \cap \mathbf{C}(\alpha)$ equals $\backslash C_2 \cap \mathbf{C}(\alpha)$ and $\backslash C_2$ is open. Hence $C_1 \cap \mathbf{C}(\alpha) = \mathrm{Cl}(C_1 \cap \mathbf{C}(\alpha)) = \mathrm{Cl}(\backslash C_2 \cap \mathbf{C}(\alpha)) = \mathrm{Cl}(\backslash C_2 \cap \mathbf{I}(\alpha))$ by Lemmas 5.1(a) and 4.7(a). Since $\backslash C_2 \cap \mathbf{I}(\alpha)$ is open, it follows by Lemma 5.1(b) that $C_1 \cap \mathbf{C}(\alpha)$ is regular-closed. Similarly, $C_2 \cap \mathbf{C}(\alpha)$ is regular-closed. So by Theorem 4.3 there are regions α_1 and α_2 such that $C_1 \cap \mathbf{C}(\alpha) = \mathbf{C}(\alpha_1)$ and $C_2 \cap \mathbf{C}(\alpha) = \mathbf{C}(\alpha_2)$. Hence $\alpha = \alpha_1 \vee \alpha_2$ by Lemma 4.3 and Theorem 4.1 and $\alpha_1 \not\!\!\infty\, \alpha_2$ by Theorem 4.7. Consequently, α is not coherent.

(b) Suppose α is not coherent, i.e. there are non-null regions α_1 and α_2 with $\alpha_1 \vee \alpha_2 = \alpha$ and $\alpha_1 \not\!\!\infty\, \alpha_2$. Then $\mathbf{C}(\alpha_1) \neq \varnothing$, $\mathbf{C}(\alpha_2) \neq \varnothing$, $\mathbf{C}(\alpha_1) \cup \mathbf{C}(\alpha_2) = \mathbf{C}(\alpha_1 \vee \alpha_2) = \mathbf{C}(\alpha)$ and $\mathbf{C}(\alpha_1) \cap \mathbf{C}(\alpha_2) = \varnothing$ by Lemma 4.4(b) and Theorems 4.6(c) and 4.7(a), respectively. I.e. $\mathbf{C}(\alpha)$ is not connected. □

LEMMA 6.4. *α is convex if and only if $\mathbf{I}(\alpha)$ is connected.*

Proof. (a) Suppose $\mathbf{I}(\alpha)$ is not connected. Then there are closed sets C_1 and C_2 such that (i) $C_1 \cap C_2 \cap \mathbf{I}(\alpha) = \varnothing$, (ii) $C_1 \cap \mathbf{I}(\alpha) \neq \varnothing$, (iii) $C_2 \cap \mathbf{I}(\alpha) \neq \varnothing$, and (iv) $\mathbf{I}(\alpha) \subseteq C_1 \cup C_2$. $C_1 \cap \mathbf{I}(\alpha)$ equals $\backslash C_2 \cap \mathbf{I}(\alpha)$ and is therefore open; hence $\mathrm{Cl}(C_1 \cap \mathbf{I}(\alpha))$ is regular-closed by Lemma 5.1(b) and equals $\mathbf{C}(\alpha_1)$ for a certain region α_1 by Theorem 4.3. Similarly $\mathrm{Cl}(C_2 \cap \mathbf{I}(\alpha))$ is regular-closed and equals $\mathbf{C}(\alpha_2)$ for some region α_2. $0 \neq \alpha_1 \leq \alpha$, $0 \neq \alpha_2 \leq \alpha$, and $\alpha_1 \not\!\!\infty\, \alpha_2$ by (ii), (iii), Lemma 4.4(a), Lemma 4.5(c) and Theorem 4.7(a). $\mathrm{Cl}(C_1 \cap \mathbf{I}(\alpha)) \cup \mathrm{Cl}(C_2 \cap \mathbf{I}(\alpha))$ equals $\mathrm{Cl}((C_1 \cup C_2) \cap \mathbf{I}(\alpha))$, which equals $\mathbf{C}(\alpha)$ by (iv). Hence $\alpha_1 \vee \alpha_2 = \alpha$ by Theorem 4.6(c) and Theorem 4.1(b). If $\alpha' \ll \alpha$, $\mathbf{C}(\alpha') \subseteq \mathbf{I}(\alpha)$ by Lemma 4.2(d). $\mathbf{C}(\alpha' \wedge \alpha_1) \subseteq \mathbf{C}(\alpha') \cap \mathrm{Cl}(C_1 \cap \mathbf{I}(\alpha)) \subseteq \mathbf{I}(\alpha) \cap C_1$, and similarly $\mathbf{C}(\alpha' \wedge \alpha_2) \subseteq \mathbf{I}(\alpha) \cap C_2$. Hence $\mathbf{C}(\alpha' \wedge \alpha_1) \cap \mathbf{C}(\alpha' \wedge \alpha_2) = \varnothing$ by (i), i.e. $(\alpha' \wedge \alpha_1) \not\!\!\infty (\alpha' \wedge \alpha_2)$. So, α is not convex. (b) Suppose α is not convex. Then there are regions $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$ with $\alpha_1 \vee \alpha_2 = \alpha$ and such that there is no region $\alpha' \ll \alpha$ with $(\alpha' \wedge \alpha_1) \infty (\alpha' \wedge \alpha_2)$. Then $\mathbf{C}(\alpha_1) \cap \mathbf{C}(\alpha_2) \cap \mathbf{I}(\alpha) = \varnothing$ by Lemma 6.2(a), $\mathbf{C}(\alpha_1) \cap \mathbf{I}(\alpha) \neq \varnothing$ and $\mathbf{C}(\alpha_2) \cap \mathbf{I}(\alpha) \neq \varnothing$ by Lemma 4.3(a) and $\mathbf{I}(\alpha) \subseteq \mathbf{C}(\alpha_1) \cup \mathbf{C}(\alpha_2)$ by Lemmas 4.2(a) and 4.6(c). Hence $\mathbf{I}(\alpha)$ is not connected. □

Next is to be shown that *continuity* of a region-based topology corresponds to *local connectedness* of the point-based topology generated by it.

DEFINITION 6.4. A topological space $\mathcal{P} = \langle P, \mathbf{C} \rangle$ is *locally connected* if and only if, given $x \in O \subseteq P$, where O is open, there exists a connected, open set A with $x \in A \subseteq O$.[15]

LEMMA 6.5. *Let $\mathcal{R}$ be a region-based topology that meets constraint* B4, *α a region of $\mathcal{R}$ and $[\nabla]_\infty$ a point of $P_\mathcal{R}$. Then*

(a) *If $[\nabla]_\infty \in \mathsf{I}(\alpha)$, there exists a convex region κ with $[\nabla]_\infty \in \mathsf{I}(\kappa)$ and $\kappa \ll \alpha$;*
(b) *If $\alpha \in \bigcap[\nabla]_\infty$ then there exists a convex region κ with $\kappa \ll \alpha$ and $\kappa \in \bigcap[\nabla]_\infty$;*
(c) *If $\alpha \neq 0$, there exists a convex region $\kappa \neq 0$ with $\kappa \leq \alpha$;*
(d) *$\alpha = \bigvee\{\kappa \mid \kappa$ is convex and $\kappa \leq \alpha\}$.*

Proof. (a) Suppose $[\nabla]_\infty \in \mathsf{I}(\alpha)$. By Lemma 4.10(a) and Theorem 4.7(b) there is a limited region β such that $[\nabla]_\infty \in \mathsf{I}(\beta)$ and $\beta \ll \alpha$. By B4 there is then a finite join $\gamma = \gamma_1 \vee \cdots \vee \gamma_n$ of convex regions γ_i, the join of any two of which not being convex, such that $\beta \leq \gamma$ and $\gamma \ll \alpha$. So $[\nabla]_\infty \in \mathsf{I}(\gamma)$. Hence by Lemma 6.2(b) $[\nabla]_\infty \in \mathsf{I}(\gamma_i)$ for one of those regions. γ_i is convex and $\gamma_i \ll \alpha$ by A4. (b) From (a) by Definition 4.1. (c) Given $\alpha \neq 0$, $\mathsf{I}(\alpha) \neq \varnothing$ by Lemma 4.4(c). Hence by (a) there exists a convex region κ with $\kappa \ll \alpha$. $\kappa \leq \alpha$ by Lemma 1.2. (d) by (c). □

So by virtue of the definitions of *continuity* and *local connectedness*, the foregoing lemmas yield the result that if region-based topology $\mathcal{R}$ is continuous then $\mathcal{P}_\mathcal{R}$ is locally connected.

THEOREM 6.5. *If $\mathcal{R}$ is continuous then $\mathcal{P}_\mathcal{R}$ is locally connected.*

Proof. Suppose $[\nabla]_\infty \in O$, where O is an open set of the point-based topology $\mathcal{P}_\mathcal{R}$. Then there exists by Lemma 4.6(b) a region α such that $[\nabla]_\infty \in \mathsf{I}(\alpha) \subseteq O$. By Lemma 6.5(a) there is then a convex region β with $[\nabla]_\infty \in \mathsf{I}(\beta) \subseteq O$ and $\beta \ll \alpha$. But $\mathsf{I}(\beta)$ is connected by Lemma 6.4. □

By the Main Theorem in Section 5 region-based topologies correspond to point-based topologies that are locally compact and satisfy T_2. It is shown now that if a point-based topology is locally connected as well, it corresponds to a continuous region-based topology.

LEMMA 6.6. *Let $\mathcal{P}$ be a locally compact, locally connected* T_2 *space. If $x \in O$, where O is open, then there is a connected, regular-open set D such that $x \in D$ and* $\mathrm{Cl}\, D \subseteq O$.

Proof. Assume $x \in O$. Then in view of Lemma 5.4 there exists a regular-closed set C with $C \subseteq O$ and $x \in \mathrm{In}\, C$. Since $\mathcal{P}$ is locally connected, there is a connected, open set A with $x \in A \subseteq \mathrm{In}\, C$. Hence $\mathrm{Cl}\, A \subseteq C$ and $x \in \mathrm{In}\, \mathrm{Cl}\, A$. So, $D = \mathrm{In}\, \mathrm{Cl}\, A$ is connected, since A is, and regular-open, and, as $\mathrm{Cl}\, D = \mathrm{Cl}\, A$, $\mathrm{Cl}\, D \subseteq O$. □

LEMMA 6.7. *Let $\mathcal{P} = \langle P, \mathbf{C} \rangle$ be locally connected and $B \subseteq P$ regular-closed and compact. Then there exists, for every regular-closed $C \subseteq P$ with $B \cap C = \varnothing$, a regular-open set T, where T is a finite union of connected regular-open sets, $B \subseteq \mathrm{Cl}\, T$, and $\mathrm{Cl}\, T \cap C = \varnothing$.*

Proof. Suppose $B \subseteq P$ is regular-closed and compact. Suppose $B \cap C = \varnothing$, where C is regular-closed. By Lemma 6.6 there exists then for every point x in B a regular-open and connected set D such that $x \in D$ and $\mathrm{Cl}\, D \cap C = \varnothing$. These open connected sets constitute a cover of B. Since B is compact, there is a finite subcover, i.e. there are sets $D_1, \ldots, D_n$ such that $B \subseteq D_1 \cup \cdots \cup D_n$. Let T be $D_1 \cup \cdots \cup D_n$. Then T is regular-open by Theorem 5.1(d), $B \subseteq T$, and $\mathrm{Cl}\, T \cap C = \varnothing$. □

By Theorem 4.4 the regular-closed sets of $\mathcal{P}_{\mathcal{R}}$ are the counterparts of the regions of $\mathcal{R}$, and by Lemma 6.4 the connected regular-open sets of $\mathcal{P}_{\mathcal{R}}$ are the counterparts of the convex regions of $\mathcal{R}$. So, given Definition 6.2 and Lemma 6.7,

THEOREM 6.6. *If $\mathcal{P}_{\mathcal{R}}$ is locally connected, $\mathcal{R}$ is continuous.*

Consequently, by the Main Theorem in Section 5, continuous region-based topologies correspond to locally compact, locally connected T_2 spaces.

I conclude this section with the observation that in a continuous region-based topology points can be identified by taking into account only limited convex regions. If follows from Lemma 6.5(b) that, if $[\nabla]_\infty$ is a point, the limited convex regions in the filter $\bigcap[\nabla]_\infty$ constitute a basis of that filter. The set $\Sigma = \{\kappa \mid \kappa$ is limited and convex and $\kappa \in \bigcap[\nabla]_\infty\}$ can be characterised by

(i) $0 \notin \Sigma$;
(ii) If $\kappa \in \Sigma$, then κ is limited and convex;
(iii) If $\kappa_1 \in \Sigma$, $\kappa_1 \leq \kappa_2$, and κ_2 is limited and convex, then $\kappa_2 \in \Sigma$;
(iv) If $\kappa_1 \in \Sigma$ and $\kappa_2 \in \Sigma$, then there exists $\kappa_3 \in \Sigma$ with $\kappa_3 \leq \kappa_1 \wedge \kappa_2$;
(v) If $\kappa_1 \in \Sigma$, then there exists $\kappa_2 \in \Sigma$ such that $\kappa_2 \ll \kappa_1$;

(vi) Σ is maximal.

In other words, Σ is a proper filter basis consisting of limited, convex regions, which is contracting and maximal. Since sets of this kinds correspond one-to-one to the families of sets of regions that have been taken to define points, they can equally well serve to identify points, provided $\mathcal{R}$ meets B4. Such an approach would be similar to Tarski's characterisation of points by means of concentric spheres[16] and also to Whitehead's characterisation of points by contracting sequences of convex volumes of space.[17]

7. Continua

This section introduces further constraints which are intended to allow the characterisation of topologies that deserve to be called *continua*. The paradign here is the linear continuum whose point-based characterisation $\mathbf{R}$ is of course well-known; all spaces $\mathbf{R}^n$, as well as their connected subspaces, are also continua. The only relevant feature which one would expect to find in point-based characterisations of all continua and which has not been considered already appears to be that of *being second countable*. The region-based counterpart of this property is B6 below. B5 is a weaker characteristic which proves to be of importance in connection with the definability of limitedness.

B5 *There is a countable set Γ of limited regions such that for every limited region α there is a region γ in Γ with $\alpha \leq \gamma$.*

B6 *There is a countable set Γ of regions such that whenever α is limited and $\alpha \ll \beta$ there is a region γ in Γ with $\alpha \ll \gamma \leq \beta$.*

The set Γ in B6 can be assumed to consist of limited regions. It is also easy to see that B6 implies B5 and that, given B5, the space of the topology is the join of countably many limited regions. A related result concerns topologies that satisfy both B4 and B6.

LEMMA 7.1. *Let $\mathcal{R}$ be a region-based topology that meets constraint* B6. *Then*

(a) *The set Γ mentioned in* B6 *can be assumed to consist exclusively of limited regions*;

(b) *$\mathcal{R}$ meets* B5.

Proof. Let Γ be a set of the kind mentioned in B6. Let Γ^* be the set of limited regions in Γ . Γ^* is a countable set of limited regions, which can play the role of Γ in B6 by A10, i.e. (a). Now suppose α is a limited region of $\mathcal{R}$. By Lemma 1.5(a) there is then a limited region β such that $\alpha \ll \beta$. By B6 there exists a region γ in Γ with $\alpha \ll \gamma \leq \beta$. γ is limited by A7, i.e. $\gamma \in \Gamma^*$, and $\alpha \leq \gamma$ by Lemma 1.2. So Γ^* is a set of the kind mentioned in B5, which is therefore satisfied by $\mathcal{R}$, i.e. (b). □

LEMMA 7.2. *Let $\mathcal{R}$ be a region-based topology that meets constraint* B5 *and Γ a set of the kind mentioned there. Then $1 = \mathrm{V}\Gamma$.*

Proof. Suppose $\mathrm{V}\Gamma \neq 1$. Then there is a region $\alpha \neq 0$ with $\alpha \wedge \gamma = 0$ for every $\gamma \in \Gamma$. By Lemma 1.3(b) α has a limited subregion β. So $\beta \wedge \gamma = 0$ for every $\gamma \in \Gamma$, which contradicts the assumption that $\mathcal{R}$ meets B5. So, $\mathrm{V}\Gamma = 1$. □

LEMMA 7.3. *Let $\mathcal{R}$ be a region-based topology that meets constraints* B4 *and* B6. *Then there is a countable set Σ of limited convex regions such that for every region $\beta \neq 0$*

(a) *there is a region $\kappa \neq 0$ in Σ with $\kappa \leq \beta$;*
(b) $\beta = \mathrm{V}\{\kappa \mid \kappa \in \Sigma$ *and* $\kappa \leq \beta\}$.

Proof. (a) By Lemma 7.1(a) the set Γ mentioned in B6 can be assumed to consist of limited regions. For every region $\gamma \neq 0$ in Γ there is a convex and limited region $\kappa \neq 0$ with $\kappa \leq \gamma$, this by Lemma 6.5(c) and A7. Let Σ be a set containing for every region $\gamma \neq 0$ in Γ a convex non-null region $\kappa \leq \gamma$. Σ is a countable set of limited convex non-null regions. Let β be any non-null region. By Lemma 1.3(a) there is a limited region $\alpha \neq 0$ with $\alpha \ll \beta$. Hence by B6 there is a region $\gamma \neq 0$ in Γ with $\gamma \leq \beta$. And so there is a region κ in Σ with $\kappa \leq \beta$. (b) By (a). □

It will be shown now that region-based topologies satisfying B5 correspond to locally compact T_2 spaces which are in addition Lindelöf spaces, while region-based topologies that meet B6 correspond to locally compact T_2 spaces that are also second countable.

DEFINITION 7.1. A topological space $\mathcal{P}$ is called a *Lindelöf space* if every open cover of the space has a countable subcover.

DEFINITION 7.2. A topological space $\mathcal{P}$ is called *second countable* if it has a countable basis.

THEOREM 7.1. *Let $\mathcal{R}$ be a region-based topology that meets constraint* B5. *Then $\mathcal{P}_{\mathcal{R}} = \langle P_{\mathcal{R}}, \mathbf{C}_{\mathcal{R}}\rangle$ is a Lindelöf space.*

Proof. Let Γ be a set of the kind mentioned in B5 and let $[\nabla]_\infty$ be any point in $P_{\mathcal{R}}$. Then, as ∇ is limited, and given Definition 4.1, there is a limited region α with $[\nabla]_\infty \in \mathbf{C}(\alpha)$. Suppose $\alpha \leq \gamma$, where $\gamma \in \Gamma$. Then $\mathbf{C}(\alpha) \subseteq \mathbf{C}(\gamma)$ by Lemma 4.2(b), and so $[\nabla]_\infty \in \mathbf{C}(\gamma)$ for some γ in Γ. Hence $P_{\mathcal{R}} = \bigcup\{\mathbf{C}(\gamma) \mid \gamma \in \Gamma\}$. Now suppose $\mathbf{S}$ is an open cover of $P_{\mathcal{R}}$. Let γ be any region in Γ. Since γ is limited, $\mathbf{C}(\gamma)$ is compact by Theorem 4.8. Hence there is a finite subset $\mathbf{S}_\gamma$ of $\mathbf{S}$ which covers $\mathbf{C}(\gamma)$. Consequently, $\bigcup\{\mathbf{S}_\gamma \mid \gamma \in \Gamma\}$, which is a subset of $\mathbf{S}$, is countable and constitutes a cover of $P_{\mathcal{R}}$. Hence $\mathcal{P}_{\mathcal{R}}$ is a Lindelöf space. □

LEMMA 7.4. *Let $\mathcal{P} = \langle P, \mathbf{C}\rangle$ be a locally compact Lindelöf space. Then there is a countable family* $\mathbf{G}$ *of regular-closed and compact sets such that* $P = \bigcup\{\operatorname{In} C \mid C \in \mathbf{G}\}$.

Proof. By Lemma 5.2 there exists for every point x of P a regular-closed and compact set D_x with $x \in \operatorname{In} D_x$. So, $\{\operatorname{In} D_x \mid x \in P\}$ constitutes an open cover of P. As $\mathcal{P}$ is a Lindelöf space, there exists a countable set $A \subseteq P$ so that $\{\operatorname{In} D_x \mid x \in A\}$ constitutes a cover of P. Let $\mathbf{G}$ be $\{D_x \mid x \in A\}$. Then $\mathbf{G}$ is a countable family of regular-closed, compact sets and $\bigcup\{\operatorname{In} C \mid C \in \mathbf{G}\} = P$. □

THEOREM 7.2. *Let $\mathcal{P}_{\mathcal{R}} = \langle P_{\mathcal{R}}, \mathbf{C}_{\mathcal{R}}\rangle$ be a Lindelöf space. Then $\mathcal{R}$ meets* B5.

Proof. By Theorem 4.9 and Lemma 7.4 there is a countable family $\mathbf{G}$ of regular-closed and compact sets with $P_{\mathcal{R}} = \bigcup\{\operatorname{In} C \mid C \in \mathbf{G}\}$. By Theorems 4.3 and 4.8 and Lemma 4.7(b) there is then a countable set Γ of limited regions such that $P_{\mathcal{R}} = \bigcup\{\mathbf{I}(\gamma) \mid \gamma \in \Gamma\}$. Let Γ^* be the closure of Γ under finite joins. If α is any limited region, $\mathbf{C}(\alpha)$ is compact by Theorem 4.8. Given that $\{\mathbf{I}(\gamma) \mid \gamma \in \Gamma\}$ constitutes an open cover of $\mathbf{C}(\alpha)$ there must be a finite subcover, i.e. there must be regions $\gamma_1, \dots, \gamma_n$ in Γ so that $\mathbf{C}(\alpha) \subseteq \mathbf{I}(\gamma_1) \cup \dots \cup \mathbf{I}(\gamma_n)$. But $\mathbf{I}(\gamma_1) \cup \dots \cup \mathbf{I}(\gamma_n) \subseteq \mathbf{I}(\gamma_1, \vee \dots \vee \gamma_n)$. Hence $\alpha \leq \gamma_1 \vee \dots \vee \gamma_n$ by Lemma 4.5(d). Since $\gamma = \gamma_1 \vee \dots \vee \gamma_n$ belongs to Γ^* and Γ^* is countable, $\mathcal{R}$ meets B5. □

So, given the Main Theorem of Section 5, the complete region-based topologies that meet B5 correspond precisely to the point-based topologies that are locally compact, T_2, and Lindelöf.

THEOREM 7.3. *Let $\mathcal{R}$ be a region-based topology that meets constraint* B6. *Then $\mathcal{P}_{\mathcal{R}}$ is second countable.*

Proof. Let Γ be a set of the kind mentioned in B6 and let $\mathbf{F}$ be $\{\mathbf{I}(\gamma) \mid \gamma \in \Gamma\}$. $\mathbf{F}$ is a countable family of open sets. Suppose that β is any region of $\mathcal{R}$ and that $[\nabla]_\infty$ is any point in $\mathbf{I}(\beta)$. Then $\beta \in \bigcap[\nabla]_\infty$ and by Lemma 3.5 there is a limited region α such that $\alpha \in \bigcap[\nabla]_\infty$ – i.e. $[\nabla]_\infty \in \mathbf{I}(\alpha)$ – and $\alpha \ll \beta$. And by B6 there is a region γ in Γ with

$\alpha \ll \gamma \leq \beta$. Hence for any point $[\nabla]_\infty \in \mathsf{I}(\beta)$ there is a region $\gamma \in \Gamma$ with $[\nabla]_\infty \in \mathsf{I}(\gamma)$ and $\gamma \leq \beta$. So $\mathsf{I}(\beta) = \bigcup\{O \mid O \in \mathbf{F} \text{ and } O \subseteq \mathsf{I}(\beta)\}$. So, every set $\mathsf{I}(\beta)$ is the union of open sets belonging to **F**. Consequently, since the family of sets $\mathsf{I}(\beta)$, i.e. the family of regular-open sets of $\mathcal{P}_\mathcal{R}$, constitutes an open basis of $\mathcal{P}_\mathcal{R}$, **F** is also an open basis of $\mathcal{P}_\mathcal{R}$. Hence $\mathcal{P}_\mathcal{R}$ is second countable. □

THEOREM 7.4. *If $\mathcal{P}_\mathcal{R}$ is second countable, $\mathcal{R}$ meets* B6.

Proof. Let **F** be a countable family of open sets such that every open set in $\mathcal{P}_\mathcal{R}$ is the union of sets in **F**. Let $\mathbf{F}^\cup$ be the closure of **F** under finite union; and let $\mathbf{F}^*$ equal $\{\operatorname{In}\operatorname{Cl} O \mid O \in \mathbf{F}^\cup\}$. By Lemma 5.1(b) every set of points in $\mathbf{F}^*$ is regular-open and hence, by Theorem 4.3(b), equals $\mathsf{I}(\gamma)$ for some region γ. Finally, let Γ be the set of these regions, i.e. $\Gamma = \{\gamma \mid \mathsf{I}(\gamma) \in \mathbf{F}^*\}$. $\mathbf{F}^\cup$, $\mathbf{F}^*$, and Γ are all countable.

Assume that α is limited and that $\alpha \ll \beta$. $\mathsf{C}(\alpha)$ is compact by Theorem 4.8 and $\mathsf{C}(\alpha) \subseteq \mathsf{I}(\beta)$ by Lemma 4.2(d). Since **F** is a basis, $\mathsf{I}(\beta) = \bigcup\{O \mid O \in \mathbf{G}\}$ for some subfamily **G** of **F**. So, **G** is an open cover of $\mathsf{C}(\alpha)$. Hence there is a finite subcover $\{O_1, \ldots, O_n\} \subseteq \mathbf{G}$ of $\mathsf{C}(\alpha)$. $\{O_1, \ldots, O_n\} \subseteq \mathbf{F}$, hence $O = O_1 \cup \cdots \cup O_n$ is a member of $\mathbf{F}^\cup$ and $\mathsf{C}(\alpha) \subseteq O$. Then $\operatorname{In}\operatorname{Cl} O \in \mathbf{F}^*$, $O \subseteq \operatorname{In}\operatorname{Cl} O$ by Lemma 5.1(b), and $\operatorname{In}\operatorname{Cl} O \subseteq \mathsf{I}(\beta)$, since $\mathsf{I}(\beta)$ is regular-open by Lemma 4.7(c). Hence $\mathsf{C}(\alpha) \subseteq \mathsf{I}(\gamma) \subseteq \mathsf{I}(\beta)$ for some region γ in Γ. Hence $\alpha \ll \gamma \leq \beta$ by Theorem 4.7(b) and Lemma 4.5(b). So, whenever α is limited and $\alpha \ll \beta$ there is a region γ in Γ with $\alpha \ll \gamma \leq \beta$. So, $\mathcal{R}$ meets B6. □

Having established the correspondence between B5 topologies and Lindelöf spaces and between B6 topologies and second countable spaces I shall define now a *continuum* as a region-based topology that is continuous, whose space is coherent and divisible, and which satisfies B6.

DEFINITION 7.3. $\mathcal{R}$ is a *continuum* if $\mathcal{R}$ is a region-based topology that meets constraints A1–A10, B2, B3, B4, and B6.

In a continuum there is, by virtue of Lemma 7.3(b), a countable set of convex regions that constitutes a basis of the Boolean algebra of regions.

In the presence of constraints B4 and B5 it can be shown that 'α is limited' is equivalent to 'If $\alpha' \leq \alpha$ and $\alpha' \between \beta$ then there is a region γ which is the join $\gamma_1 \vee \cdots \vee \gamma_n$ of finitely many convex regions $\gamma_1, \ldots, \gamma_n$ such that $\alpha' \leq \gamma$ and $\gamma \between \beta$'. This result will now be proved.

Consider then a region-based topology $\mathcal{R}$ that meets B5 and an unlimited region τ of $\mathcal{R}$. Let Γ be a countable set of limited regions of the kind mentioned in B5. Then $1 = \bigvee\Gamma$ by Lemma 7.2. Since 1 is unlimited by A7, Γ is denumerably infinite by A8, i.e. $\Gamma = \{\gamma_1, \ldots, \gamma_n, \ldots\}$.

Given this set, define a set $\Sigma = \{\sigma_1, \ldots, \sigma_n, \ldots\}$ of regions as follows: (i) $\sigma_1 = \gamma_1$. (ii) Having defined the limited region σ_i, note that by A8 and Lemma 1.5(c) there exists a limited region σ such that both $\sigma_i \vee \gamma_{i+1} \ll \sigma$ and $\sigma_i \vee \gamma_{i+1} < \sigma$. Let σ_{i+1} be such a σ. As a consequence (a) both $\sigma_i \ll \sigma_j$ and $\sigma_i < \sigma_j$ for $i < j$; (b) $\mathrm{V}\Sigma = 1$; and since by B5 $\alpha \leq \gamma_n$, for some n, when α is limited, (c) $\alpha \leq \sigma_n$ for some n, when α is limited.

Next define a set $\{\sigma_1^*, \ldots, \sigma_n^*, \ldots\}$ as follows: $\sigma_1^* = \sigma_1$ and, for $n \geq 2$, $\sigma_n^* = \sigma_n \wedge -\sigma_{n-1}$. Then $1 = \mathrm{V}\{\sigma_1^*, \ldots, \sigma_n^*, \ldots\}$, $\sigma_n^* \neq 0$ and σ_n^* is limited for every n, and $\sigma_m^* \not\infty \sigma_n^*$ when $|m - n| > 1$. Because of A8 the set $\{\sigma_1^* \wedge \tau, \ldots, \sigma_n^* \wedge \tau, \ldots\}$ must have denumerably many non-null members $\sigma_{n_1}^* \wedge \tau, \ldots, \sigma_{n_i}^* \wedge \tau, \ldots$. Given these, define two further sets of regions $\{\alpha_1, \ldots, \alpha_i, \ldots\}$ and $\{\beta_1, \ldots, \beta_i, \ldots\}$ as follows: $\alpha_1 = \sigma_{n_1}^* \wedge \tau$, $\beta_1 = \sigma_{n_3}^*$; $\alpha_2 = \sigma_{n_5}^* \wedge \tau$, $\beta_2 = \sigma_{n_7}^*; \ldots; \alpha_i = \sigma_{n_{4i-3}}^* \wedge \tau$; $\beta_i = \sigma_{n_{4i-1}}^*, \ldots$. Then $\alpha_i \not\infty \beta_j$ for every i and j, while $\alpha_i \not\infty \alpha_j$ and $\beta_i \not\infty \beta_j$ for $i \neq j$. Let α be $\mathrm{V}\{\alpha_1, \ldots, \alpha_n, \ldots\}$ and β be $\mathrm{V}\{\beta_1, \ldots, \beta_n, \ldots\}$. Then $\alpha \leq \tau$. Now assume that $\alpha \not\infty \beta$. Then by A9 there are limited regions $\alpha' \leq \alpha$ and $\beta' \leq \beta$ with $\alpha' \not\infty \beta'$. Hence $\alpha' \leq \sigma_i$ and $\beta' \leq \sigma_j$ for some i and j. Let m be the smallest number such that both $i \leq n_{4m}$ and $j \leq n_{4m}$. Then $\alpha' \leq \alpha_1 \vee \cdots \vee \alpha_m$ and $\beta' \leq \beta_1 \vee \cdots \vee \beta_m$, and hence $\alpha' \not\infty \beta'$ by A5 and the definition of α_i and β_j. Consequently, we must reject the assumption that $\alpha \infty \beta$ and conclude that $\alpha \not\infty \beta$.

Now suppose $\gamma_1, \ldots, \gamma_m$ are convex regions, $\gamma = \gamma_1 \vee \cdots \vee \gamma_m$, and $\alpha \leq \gamma$. Since α is the join of denumerably many regions, there is at least one region γ_k, $1 \leq k \leq m$, which overlaps with at least two of the regions $\alpha_1, \alpha_2, \ldots$; say $\gamma_k \wedge \alpha_i \neq 0$ and $\gamma_k \wedge \alpha_j \neq 0$, where $i < j$; i.e. $\gamma_k \wedge (\sigma_{n_{4i-3}}^* \wedge \tau) \neq 0$ and $\gamma_k \wedge (\sigma_{n_{4j-3}}^* \wedge \tau) \neq 0$. In order to show that $\gamma_k \wedge \beta_i \neq 0$, let γ' be $\gamma_k \wedge \sigma_{n_{4i-1}-1}$ and let γ'' be $\gamma_k \wedge -\sigma_{n_{4i-1}}$. $\gamma' \neq 0$, since $\gamma_k \wedge \sigma_{n_{4i-3}} \neq 0$ and $\sigma_{n_{4i-3}} < \sigma_{n_{4i-1}-1}$; $\gamma'' \neq 0$, since $\gamma_k \wedge -\sigma_{n_{4j-3}-1} \neq 0$ and $-\sigma_{n_{4j-3}-1} < -\sigma_{n_{4i-1}}$; and $\gamma' \not\infty \gamma''$ since $\sigma_{n_{4i-1}-1} \ll \sigma_{n_{4i-1}}$. If now $\gamma_k \wedge \beta_i = 0$, then $\gamma' \vee \gamma'' = \gamma_k \wedge (\sigma_{n_{4i-1}-1} \vee -\sigma_{n_{4i-1}}) = \gamma_k \wedge -\sigma_{n_{4i-1}}^* = \gamma_k \wedge -\beta_i = \gamma_k$. But in that case γ_k is not coherent, hence not convex, contrary to the assumption. So $\gamma_k \wedge \beta_i \neq 0$. Hence $\gamma \wedge \beta \neq 0$, and hence $\gamma \infty \beta$ by Lemma 1.1. Hence

THEOREM 7.5. *Let $\mathcal{R}$ meet* B5 *and let τ be an unlimited region of $\mathcal{R}$. Then there exists a region $\alpha \leq \tau$ and a region β such that $\alpha \not\infty \beta$, but, for every region $\gamma = \gamma_1 \vee \cdots \vee \gamma_n$, where each of $\gamma_1, \ldots, \gamma_n$ is convex, if $\alpha \leq \gamma$, then $\gamma \infty \beta$.*

On the other hand, if a region-based topology meets constraint B4, then the converse of Theorem 7.5 holds.

THEOREM 7.6. *Let $\mathcal{R}$ meet constraint* B4 *and let τ be a limited region of $\mathcal{R}$. Then there exists for every $\alpha \leq \tau$ and any region β with $\alpha \not\infty \beta$ a region γ which is a finite join of convex regions such that $\alpha \leq \gamma$ and $\gamma \not\infty \beta$.*

Proof by constraint A7. □

Consequently, *limitedness* can be defined for any region-based topology that is continuous and satisfies B5; in particular, limitedness can be defined for any continuum.[18]

DEFINITION 7.4. τ is *limited* if and only if for every $\alpha \leq \tau$ and any β such that $\alpha \not\infty \beta$ there exists a region γ which is a finite join of convex regions such that $\alpha \leq \gamma$ and $\gamma \not\infty \beta$.

Given this definition, A6, A7, and A8 are immediate consequences and can therefore be dispensed with. In particular, continua can be characterised as complete Boolean algebras on which a relation ∞ is defined that meets A1–A5, A9, A10, B2, B3, and B6. The definitions invoked in this characterisation are Definition 1.1, Definition 1.2, Definition 1.3, and Definition 7.4.

Bringing the Main Theorem of Section 5 together with Theorems 6.4, 6.5 and 6.6, and Theorems 7.3 and 7.4 we have the result that region-based topologies that are continua correspond to point-based topologies that have more than one point, are connected, locally connected, locally compact T_2 spaces, and have a countable basis.

8. Functions

Point-based topology deals with functions that associate with every point in the space of one topology a point in a second topology. Since the spaces of region-based topologies are, in general, non-atomic, the ordinary notion of a function is not directly applicable. What is required are functions from regions to regions which conform to the mereological structure of the domain so as to be recognisable as counterparts of first-order functions. For example, when α is a subregion of β, then α must be mapped onto a subregion of the region that β is mapped onto. In fact, functions on Ω are introduced by analogy with the *images* of *sets of points* (under a first-order function).[19]

DEFINITION 8.1. Let Ω and Ω^* be Boolean algebras. Then a function ϕ from Ω into Ω^* is a *mereological mapping* if ϕ satisfies these constraints:

(i) $\phi(\alpha) = 0^*$ if and only if $\alpha = 0$;
(ii) If $\alpha \leq \beta$ then $\phi(\alpha) \leq \phi(\beta)$;
(iii) If $0^* \neq \alpha^* \leq \phi(\beta)$ for $\alpha^* \in \Omega^*$, then there exists a region α in Ω with $0 \neq \alpha \leq \beta$ and $\phi(\alpha) \leq \alpha^*$.

The next lemmas deal with immediate consequences of this definition. Again, all Boolean algebras are assumed to be complete.

LEMMA 8.1. *Let ϕ be a mereological mapping from Ω to Ω^*. Then*

(a) $\phi(\alpha \wedge \beta) \leq \phi(\alpha) \wedge \phi(\beta)$;
(b) $\phi(\alpha \vee \beta) = \phi(\alpha) \vee \phi(\beta)$; *and*
(c) $\phi(\mathrm{V}\Gamma) = \mathrm{V}\{\phi(\alpha) \mid \alpha \in \Gamma\}$.

Proof. (a) By clause (ii) of Definition 8.1. (b) $\phi(\alpha)\vee\phi(\beta) \leq \phi(\alpha\vee\beta)$ by clause (ii) of Definition 8.1. Suppose that $\phi(\alpha) \vee \phi(\beta) < \phi(\alpha \vee \beta)$. Then there is a region γ^* of Ω^* with $\gamma^* \neq 0$, $\gamma^* \leq \phi(\alpha \vee \beta)$, and both $\gamma^* \wedge \phi(\alpha) = 0$ and $\gamma^* \wedge \phi(\beta) = 0$. Hence by clause (iii) there exists in Ω a region γ with $\gamma \neq 0$, $\gamma \leq \alpha \vee \beta$ and $\phi(\gamma) \leq \gamma^*$. But then there exists $\delta \neq 0$ such that $\phi(\delta) \leq \gamma^*$ and either $\delta \leq \alpha$ or $\delta \leq \beta$. By clause (ii) then either $\phi(\delta) \leq \phi(\alpha)$ or $\phi(\delta) \leq \phi(\beta)$, both of which are impossible. So, $\phi(\alpha) \vee \phi(\beta) = \phi(\alpha \vee \beta)$. (c) Analogously. □

LEMMA 8.2. *Let ϕ be a mereological mapping from Ω to Ω^* and let α^* be a region in Ω^* with $\alpha^* \neq 0$ and $\alpha^* \leq \phi(1)$. Then $\alpha^* = \phi(\mathrm{V}\Gamma)$, where Γ is $\{\beta \mid \beta \in \Omega$ and $\phi(\beta) \leq \alpha^*\}$.*

Proof. Assume $\phi(\mathrm{V}\Gamma) \neq \alpha^*$, i.e., by Lemma 8.1(c) and clause (ii) of Definition 8.1, $\phi(\mathrm{V}\Gamma) < \alpha^*$. Let β^* be $\alpha^* \wedge -\phi(\mathrm{V}\Gamma)$. Then $\beta^* \neq 0$ and $\beta^* \leq \alpha^* \leq \phi(1)$. By clause (iii) there exists a region β in Ω such that $0 \neq \beta \leq 1$ and $\phi(\beta) \leq \beta^*$. Hence $0 \neq \phi(\beta) \leq \alpha^*$, and $\beta \in \mathrm{V}\Gamma$. Consequently, $\phi(\beta) \leq \phi(\mathrm{V}\Gamma)$, which is impossible since $\phi(\beta) \leq \beta^*$ and $\beta^* \wedge \phi(\mathrm{V}\Gamma) = 0$. □

This result permits definition of the inverse mapping ϕ^{-1} of a mereological mapping ϕ.

DEFINITION 8.2. Let ϕ be a mereological mapping from Ω to Ω^*. Then ϕ^{-1} is this mapping from Ω^* to Ω:

$$\phi^{-1}(\alpha^*) = \mathrm{V}\{\beta \mid \beta \in \Omega \text{ and } \phi(\beta) \leq \alpha^*\}.$$

By virtue of this definition and Lemma 8.2 we have.

LEMMA 8.3. *Let ϕ be a mereological mapping from Ω to Ω^* and let α^* be a region in Ω^*. Then*

(a) $\phi(\phi^{-1}(\alpha^*)) \leq \alpha^*$;
(b) *If* $\alpha^* \leq \phi(1)$, *then* $\phi(\phi^{-1}(\alpha^*)) = \alpha^*$.

The inverse mapping of a mereological mapping need not itself be a mereological mapping.

LEMMA 8.4. *Let* ϕ *be a mereological mapping from* Ω *to* Ω^* *and let* α *be in* Ω *and* β^* *in* Ω^*. *Then*

(a) $\phi(\alpha \wedge \phi^{-1}(\beta^*)) = \phi(\alpha) \wedge \beta^*$;
(b) *If* $\phi(\alpha) = \beta^* \vee \gamma^*$, *then* $\alpha = (\alpha \wedge \phi^{-1}(\beta^*)) \vee (\alpha \wedge \phi^{-1}(\gamma^*))$.

Proof. (a) Obviously, $\phi(\alpha \wedge \phi^{-1}(\beta^*)) \leq \phi(\alpha) \wedge \beta^*$ by Lemma 8.1(a) and Lemma 8.3(a). If $\phi(\alpha) \wedge \beta^* = 0^*$, it follows that $\phi(\alpha \wedge \phi^{-1}(\beta^*)) = \phi(\alpha) \wedge \beta^*$. So, assume $\phi(\alpha) \wedge \beta^* \neq 0^*$ and suppose there exists a region $\gamma^* \neq 0^*$ in Ω^* such that $\gamma^* \leq \phi(\alpha) \wedge \beta^*$, but $\gamma^* \wedge \phi(\alpha \wedge \phi^{-1}(\beta^*)) = 0^*$. By clause (iii) of Definition 8.1 there exists a region $\gamma \neq 0$ with $\gamma \leq \alpha$ and $\phi(\gamma) \leq \gamma^*$. Hence $\phi(\gamma) \leq \beta^*$, and so $\gamma \leq \phi^{-1}(\beta^*)$ by the definition of ϕ^{-1}. Hence $\phi(\gamma) \neq 0^*$ by clause (i) and $\phi(\gamma) \leq \phi(\alpha \wedge \phi^{-1}(\beta^*))$ by clause (ii). This contradicts the supposition. So $\phi(\alpha \wedge \phi^{-1}(\beta^*)) = \phi(\alpha) \wedge \beta^*$. (b) Suppose $\phi(\alpha) = \beta^* \vee \gamma^*$. Let δ be any region with $\delta \neq 0$ and $\delta \leq \alpha$. It follows by clause (ii) that $\phi(\delta) \wedge \beta^* \neq 0^*$ or $\phi(\delta) \wedge \gamma^* \neq 0^*$. Hence by (a) and clause (i) either $\delta \wedge \phi^{-1}(\beta^*) \neq 0$ or $\delta \wedge \phi^{-1}(\gamma^*) \neq 0$, i.e. $\delta \wedge (\phi^{-1}(\beta^*) \vee \phi^{-1}(\gamma^*)) \neq 0$. Consequently, $\alpha = (\alpha \wedge \phi^{-1}(\beta^*)) \vee (\alpha \wedge \phi^{-1}(\gamma^*))$. □

The next lemmas throw light on the fate of ultrafilters under mereological mappings.

LEMMA 8.5. *Let* ϕ *be a mereological mapping from* Ω *to* Ω^* *and let* ∇ *be an ultrafilter on* Ω. *Then* $\{\phi(\alpha) \mid \alpha \in \nabla\}$ *is an ultrafilter basis on* Ω^*.

Proof. Let ∇^* be the set $\{\alpha^* \mid \phi(\alpha) \leq \alpha^*$ for some α in $\nabla\}$. ∇^* is a filter. For $\phi(\alpha \wedge \beta) \leq \phi(\alpha) \wedge \phi(\beta)$ by Lemma 8.1(a). Suppose ∇^* is not maximal. Then there is a region β^* in Ω^* such that $\beta^* \notin \nabla^*$, while $\beta^* \wedge \alpha^* \neq 0$ for all regions α^* in ∇^*. Now let α be any member of ∇. Then $\beta^* \wedge \phi(\alpha) \neq 0$. Hence $\phi^{-1}(\beta^*) \wedge \alpha \neq 0$ by Lemma 8.4(a). So $\phi^{-1}(\beta^*) \wedge \alpha \neq 0$ for every member α of ∇, while $\phi^{-1}(\beta^*) \notin \nabla$ by Lemma 8.3(a). But then, contrary to the assumption, ∇ is not an ultrafilter. So ∇^* is an ultrafilter, too, and $\{\phi(\alpha) \mid \alpha \in \nabla\}$, which generates ∇^*, is an ultrafilter basis. □

So, in effect, a mereological mapping takes ultrafilters into ultrafilters, and this justifies extending the definition of the mereological mapping ϕ to ultrafilters.

DEFINITION 8.3. Let ϕ a mereological mapping from Ω to Ω^*. Then, for any ultrafilter ∇ on Ω, $\phi(\nabla)$ is the ultrafilter on Ω^* generated by $\{\phi(\alpha) \mid \alpha \in \nabla\}$.

Of course, not every function from ultrafilters on Ω to ultrafilters on Ω^* amounts to a mereological mapping.

Now assume that the Boolean algebras Ω and Ω^* consist of the regions of region-based topologies $\mathcal{R}$ and $\mathcal{R}^*$ so that we can consider mereological mappings from $\mathcal{R}$ to $\mathcal{R}^*$, i.e. from the Boolean algebra of $\mathcal{R}$ to that of $\mathcal{R}^*$. Definition 8.1 makes no reference to topological structure. Every mereological mapping from Ω to Ω^* makes sense as a function from $\mathcal{R} = \langle\Omega, \infty, \Delta\rangle$ to $\mathcal{R}^* = \langle\Omega^*, \infty^*, \Delta^*\rangle$, irrespective of the topological structure encapsulated in ∞ and Δ, and ∞^* and Δ^*, respectively. But the structure may simplify specification of the function. Firstly, in every region-based topology any mereological mapping ϕ is completely determined by its values $\phi(\beta)$ for limited regions β. Secondly, if $\mathcal{R}$ is a continuous topology, any mereological mapping ϕ is completely determined by its values $\phi(\kappa)$ for limited convex regions κ. Thirdly, if $\mathcal{R}$ is a continuum then there is a countable set Σ of limited and convex regions such that every mereological mapping ϕ is completely determined by its values $\phi(\sigma)$ for regions σ in Σ.

THEOREM 8.1. *Let ϕ be a mereological mapping from $\mathcal{R}$ to $\mathcal{R}^*$.*

(a) (1) *Let ϕ^1 be the restriction of ϕ to limited regions in Ω. Then* $\phi(\alpha) = \bigvee\{\phi^1(\beta) \mid \beta \text{ is limited and } \beta \leq \alpha\}$ *and ϕ^1 meets the restrictions* $(\mathrm{i})^1$, $(\mathrm{ii})^1$, *and* $(\mathrm{iii})^1$ *of clauses* (i), (ii), *and* (iii) *of Definition* 8.1 *to limited regions of* Ω. (2) *If ϕ^1 is a function from the limited regions of Ω to Ω^* which satisfies* $(\mathrm{i})^1$, $(\mathrm{ii})^1$, *and* $(\mathrm{iii})^1$ *then the extension* $\phi(\alpha) = \bigvee\{\phi^1(\beta) \mid \beta \text{ is limited and } \beta \leq \alpha\}$ *to all regions of Ω is a mereological mapping.*

(b) *Assume that $\mathcal{R}$ is continuous.* (1) *Let ϕ^c be the restriction of ϕ to limited convex regions in Ω. Then* $\phi(\alpha) = \bigvee\{\phi^c(\kappa) \mid \kappa \text{ is limited and convex and } \kappa \leq \alpha\}$ *and ϕ^c meets the restrictions* $(\mathrm{i})^c$, $(\mathrm{ii})^c$, *and* $(\mathrm{iii})^c$ *of clauses* (i), (ii), *and* (iii) *of Definition* 8.1 *to limited convex regions of* Ω. (2) *If ϕ^c is a function from the limited convex regions of Ω to Ω^* which satisfies* $(\mathrm{i})^c$, $(\mathrm{ii})^c$, *and* $(\mathrm{iii})^c$ *then the extension* $\phi(\alpha) = \bigvee\{\phi^c(\kappa) \mid \kappa \text{ is limited and convex and } \kappa \leq \alpha\}$ *to all regions of Ω is a mereological mapping.*

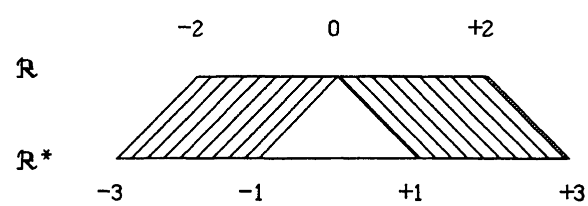
(c) *Assume that $\mathcal{R}$ is a continuum. Then there is a countable set Σ of limited convex regions of Ω for which the following holds good.* (1) *If ϕ is a mereological mapping and ϕ^s its restriction to the regions in Σ, then* $\phi(\alpha) = \bigvee\{\phi^s(\sigma) \mid \sigma \in \Sigma \text{ and } \sigma \leq \alpha\}$ *and ϕ^s meets the restrictions* $(\mathrm{i})^s$, $(\mathrm{ii})^s$, *and* $(\mathrm{iii})^s$, *of clauses* (i), (ii), *and* (iii) *of Definition* 8.1 *to the regions* *in* Σ. (2) *If* ϕ^s *is a function from the regions in* Σ *to* Ω^* *which satisfies* (i)^s, (ii)^s, *and* (iii)^s, *then the extension* $\phi(\alpha) = \mathrm{V}\{\phi^s(\sigma) \mid \sigma \in \Sigma \text{ and } \sigma \leq \alpha\}$ *to all regions of* Ω *is a mereological mapping.*

Proof. (a) (1) That ϕ can be recovered from ϕ^1 follows from Lemmas 1.3(c) and 8.1(c). As to ϕ^1, only (iii)^1 requires attention, and it follows from Lemma 1.3(b). (2) ϕ meets (i) since 0 is limited by A6. (ii) is trivial, and as to (iii), assume that $0^* \neq \alpha^* \leq \phi(\beta)$. Since $\phi(\beta) = \mathrm{V}\{\phi^1(\gamma) \mid \gamma \text{ is limited and } \gamma \leq \beta\}$ there must be a limited region $\gamma \neq 0$ with $\gamma \leq \beta$ and $\phi^1(\gamma) \wedge \alpha^* \neq 0^*$. Hence by (iii)^1 there is a limited region $\delta \leq \gamma$ such that $\delta \neq 0$ and $\phi^1(\delta) \leq \phi^1(\gamma) \wedge \alpha^*$. But $\phi(\delta) = \phi^1(\delta)$ so that $\phi(\delta) \leq \alpha^*$ and, since $\delta \leq \beta$, (iii).

(b) (1) That ϕ can be recovered from ϕ^c follows from Lemmas 1.3(c), 6.5(d) and 8.1(c). As to ϕ^c meeting (i)^c, (ii)^c, and (iii)^c, only (iii)^c needs proof; it follows from Lemma 6.5(c) and A7. (2) ϕ meets (i) since 0 is convex by Lemma 1.7. (ii) is trivial and as to (iii), assume that $0^* \neq \alpha^* \leq \phi(\beta)$. Since $\phi(\beta) = \mathrm{V}\{\phi^c(\rho) \mid \rho \text{ is limited and convex and } \rho \leq \beta\}$ there must be a limited and convex region $\rho \neq 0$ with $\rho \leq \beta$ and $\phi^c(\rho) \wedge \alpha^* \neq 0^*$. Hence by (iii)^c there is a limited, convex region $\kappa \leq \rho$ such that $\kappa \neq 0$ and $\phi^c(\kappa) \leq \phi^c(\rho) \wedge \alpha^*$. But $\phi(\kappa) = \phi^c(\kappa)$ so that $\phi(\kappa) \leq \alpha^*$ and, since $\kappa \leq \beta$, (iii).

(c) By Lemma 7.3(b) there is a countable set Γ of limited convex regions of Ω such that, for any region $\alpha \neq 0$ of Ω, $\alpha = \mathrm{V}\{\gamma \mid \gamma \in \Gamma \text{ and } \gamma \leq \alpha\}$. Let Σ be the set $\Gamma \cup \{0\}$. (1) Let ϕ^s be the restriction of ϕ to the regions in Σ. By Lemma 8.1(c) $\phi(\alpha) = \mathrm{V}\{\phi(\sigma) \mid \sigma \in \Sigma \text{ and } \sigma \leq \alpha\}$. ϕ^s obviously meets (i)^s and (ii)^s; that it meets (iii)^s follows from Lemma 7.3(a). (2) ϕ meets (i), since $0 \in \Sigma$. (ii) is obvious, and as to (iii), assume that $0^* \neq \alpha^* \leq \phi(\beta)$. Since $\phi(\beta) = \mathrm{V}\{\phi^s(\rho) \mid \rho \in \Sigma \text{ and } \rho \leq \beta\}$ there must be a region $\rho \in \Sigma$ with $\rho \neq 0$, $\rho \leq \beta$ and $\phi^s(\rho) \wedge \alpha^* \neq 0^*$. Hence by (iii)^s there is a region $\sigma \in \Sigma$ such that $\sigma \leq \rho$ and $\phi^s(\sigma) \leq \phi^s(\rho) \wedge \alpha^*$. But $\phi(\sigma) = \phi^s(\sigma)$ so that $\phi(\sigma) \leq \alpha^*$ and hence (iii). □

Mereological mappings generally do not correspond to point-functions in the point-based counterpart of the region-based topology, as the following example shows.



The mapping in the example takes the region limited by -2 and 0 into the region limited by -3 and -1 and the region limited by 0 and $+2$ into the region limited by $+1$ and $+3$. There is no point in $\mathcal{R}^*$ which could be identified as the image of 0 in $\mathcal{R}$. This is where the difference between point-based and region-based topology is most pronounced.

However, for the subclass of *continuous* and *bounded* mereological mappings there is correspondence with functions from points to points. These notions are easily defined. A *continuous* mapping is a function that preserves the relation of connection and a *bounded* mapping one that preserves limitedness.

DEFINITION 8.4. A mereological mapping ϕ from $\mathcal{R}$ to $\mathcal{R}^*$ is *continuous* if $\phi(\alpha) \infty^* \phi(\beta)$ when $\alpha \infty \beta$.

DEFINITION 8.5. A mereological mapping ϕ from $\mathcal{R}$ to $\mathcal{R}^*$ is *bounded* if $\phi(\alpha)$ is limited in Ω^* when α is limited in Ω.

Again, it is sufficient for a mereological mapping ϕ to be continuous that the restriction ϕ^1 of ϕ to limited regions of Ω be continuous; and, trivially, ϕ is bounded when ϕ^1 is bounded.

LEMMA 8.6. *Let ϕ^1 be a mereological mapping from the limited regions of $\mathcal{R}$ to $\mathcal{R}^*$ such that $\phi^1(\alpha) \infty^* \phi^1(\beta)$ if $\alpha \infty \beta$. Then the extension ϕ of ϕ^1 to all regions of $\mathcal{R}$ is continuous.*

Proof by A9. □

As one would expect, continuous mereological mappings take coherent regions into coherent regions.

LEMMA 8.7. *Let ϕ be a continuous mereological mapping from $\mathcal{R}$ to $\mathcal{R}^*$. Then if $\kappa \in \Omega$ is coherent, $\phi(\kappa)$ is coherent.*

Proof. Suppose $\phi(\kappa)$ is not coherent, i.e. $\phi(\kappa) = \alpha^* \vee \beta^*$, where $\alpha^* \neq 0^*$, $\beta^* \neq 0^*$ and $\alpha^* \not\infty^* \beta^*$. Then, by Lemma 8.4(b) $\kappa = (\kappa \wedge \phi^{-1}(\alpha^*)) \vee (\kappa \wedge \phi^{-1}(\beta^*))$. Since by Lemma 8.4(a) $\phi(\kappa \wedge \phi^{-1}(\alpha^*)) = \alpha^*$ and $\phi(\kappa \wedge \phi^{-1}(\beta^*)) = \beta^*$, $(\kappa \wedge \phi^{-1}(\alpha^*)) \neq 0$ and $(\kappa \wedge \phi^{-1}(\beta')) \neq 0$ by clause (i) of Definition 8.1 and $(\kappa \wedge \phi^{-1}(\alpha^*)) \not\infty^* (\kappa \wedge \phi^{-1}(\beta^*))$ by virtue of ϕ's being a continuous mapping, i.e. κ is not coherent. □

Given Definition 8.3, which extends mereological mappings to ultrafilters, it can be shown that continuous mappings preserve the relation of connection between ultrafilters and that bounded mappings take limited ultrafilters into limited ultrafilters.

LEMMA 8.8. *Let ϕ be a mereological mapping from $\mathcal{R}$ to $\mathcal{R}^*$, and let ∇_1 and ∇_2 be ultrafilters on Ω. Then ϕ is continuous if and only if $\phi(\nabla_1) \infty^* \phi(\nabla_2)$ whenever $\nabla_1 \infty \nabla_2$.*

Proof. (a) Suppose ϕ is continuous and assume $\nabla_1 \infty \nabla_2$. Let α_1^* belong to $\phi(\nabla_1)$ and α_2^* to $\phi(\nabla_2)$. By Definition 8.3 there is a region $\alpha_1 \in \nabla_1$ with $\phi(\alpha_1) \leq \alpha_1^*$ and a region $\alpha_2 \in \nabla_2$ with $\phi(\alpha_2) \leq \alpha_2^*$. $\alpha_1 \infty \alpha_2$ by the assumption. Hence $\phi(\alpha_1) \infty^* \phi(\alpha_2)$, as ϕ is continuous, and $\alpha_1^* \infty^* \alpha_2^*$ by A4 and Lemma 1.1. So, $\phi(\nabla_1) \infty^* \phi(\nabla_2)$. (b) Suppose ϕ is not continuous and, in particular, $\alpha \infty \beta$ but $\phi(\alpha) \not\infty^* \phi(\beta)$. By Lemma 2.8 there are limited ultrafilters ∇_α and ∇_β on Ω with $\alpha \in \nabla_\alpha$, $\beta \in \nabla_\beta$ and $\nabla_\alpha \infty \nabla_\beta$. But $\phi(\nabla_\alpha) \not\infty^* \phi(\nabla_\beta)$, since $\phi(\alpha) \in \phi(\nabla_\alpha)$ and $\phi(\beta) \in (\nabla_\beta)$. □

LEMMA 8.9. *Let ϕ be a mereological mapping from $\mathcal{R}$ to $\mathcal{R}^*$, and let ∇ be an ultrafilter on Ω. Then ϕ is bounded if and only if $\phi(\nabla)$ is limited whenever ∇ is limited.*

Proof. (a) Obviously, if ϕ is bounded, then $\phi(\nabla)$ is limited if ∇ is. (b) Suppose that ϕ is not bounded and, in particular, that α is limited (i.e. $\alpha \in \Delta$) but $\phi(\alpha)$ is not (i.e. $\phi(\alpha) \notin \Delta^*$). The set $\Sigma = \{\beta \mid \phi(\alpha \wedge -\beta) \in \Delta^*\}$ is a filter on Ω. Let γ be any region in Ω. If $\phi(\gamma) \in \Delta^*$ then $\phi(\alpha \wedge \gamma) \in \Delta^*$ by Lemma 8.1(a) and A7, and hence $-\gamma \in \Sigma$. So if $\gamma \wedge \beta \neq 0$ for every $\beta \in \Sigma$ then $\phi(\gamma) \notin \Delta^*$. Hence if ∇ is an ultrafilter with $\Sigma \subseteq \nabla$ then $\phi(\nabla)$ is unlimited. □

So, in the light of the last two results, a mereological mapping ϕ which is both continuous and bounded effectively maps the point $[\nabla]_\infty$ onto the point $[\phi(\nabla)]_{\infty^*}$. Hence, a continuous and bounded mereological mapping from $\mathcal{R}$ to $\mathcal{R}^*$ allows definition of a point-function from (the space $P_\mathcal{R}$ of) $\mathcal{P}_\mathcal{R}$ to (the space $P_{\mathcal{R}^*}$ of) $\mathcal{P}_{\mathcal{R}^*}$.

DEFINITION 8.6. Let $\mathcal{R} = \langle \Omega, \infty, \Delta \rangle$ and $\mathcal{R}^* = \langle \Omega^*, \infty^*, \Delta^* \rangle$ be region-based topologies and ϕ a continuous and bounded mereological mapping from Ω to Ω^*. Let $\mathcal{P}_\mathcal{R} = \langle P_\mathcal{R}, \mathbf{C}_\mathcal{R} \rangle$ and $\mathcal{P}_{\mathcal{R}^*} = \langle P_{\mathcal{R}^*}, \mathbf{C}_{\mathcal{R}^*} \rangle$ be the point-based topologies generated by $\mathcal{R}$ and $\mathcal{R}^*$, respectively. Then f_ϕ is this function from $P_\mathcal{R}$ to $P_{\mathcal{R}^*}$, where $[\nabla]_\infty$ is any point of $P_\mathcal{R}$:

$$f_\phi([\nabla]_\infty) = [\phi(\nabla)]_{\infty^*}.$$

On the other hand, Lemmas 8.8 and 8.9 also show that if a mereological mapping ϕ from $\mathcal{R}$ to $\mathcal{R}^*$ is not both continuous and bounded then ϕ does not correspond to any function from $\mathcal{P}_\mathcal{R}$ to $\mathcal{P}_{\mathcal{R}^*}$, since there will be points in $\mathcal{P}_\mathcal{R}$ that have no image under ϕ or not a single image under ϕ. Consequently, THEOREM 8.2. *A mereological mapping ϕ from $\mathcal{R}$ to $\mathcal{R}^*$ gives rise to a function from $\mathcal{P}_{\mathcal{R}}$ to $\mathcal{P}_{\mathcal{R}^*}$ if and only if ϕ is both continuous and bounded.*

The next question to ask is what kind of a point-function f_ϕ a continuous and bounded mereological mapping ϕ gives rise to. The first result is that f_ϕ is continuous in the point set theoretical sense.

DEFINITION 8.7. Let $\mathcal{P} = \langle P, \mathbf{C}\rangle$ and $\mathcal{P}^* = \langle P^*, \mathbf{C}^*\rangle$ be point-based topologies and f a function from P to P^*. Then f is *continuous at a point* x of P if for every open set O^* of P^* with $f(x) \in O^*$ there is an open set O of P with $x \in O$ and $f(O) \subseteq O^*$. f is *continuous* if f is continuous at every point x of P.

LEMMA 8.10. *Let ϕ be a continuous and bounded mereological mapping from $\mathcal{R}$ to $\mathcal{R}^*$. Then*

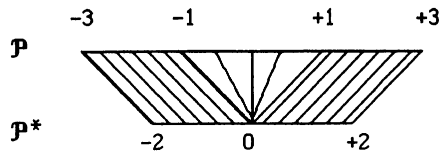
(a) $f_\phi(\mathbf{C}(\alpha)) \subseteq \mathbf{C}(\phi(\alpha))$;
(b) $f_\phi(\mathbf{I}(\alpha)) \subseteq \mathbf{C}(\phi(\alpha))$.

Proof. (a) Suppose $[\nabla^*]_{\infty^*} \in f_\phi(\mathbf{C}(\alpha))$. Then by Definitions 8.6 and 4.1 $[\nabla^*]_{\infty^*}$ is $[\phi(\nabla)]_{\infty^*}$ for some ∇ with $\alpha \in \bigcup[\nabla]_\infty$. Since $\phi(\beta) \in \phi(\nabla')$ if $\beta \in \nabla'$ by Definition 8.3, $\phi(\alpha) \in \bigcup[\phi(\nabla)]_{\infty^*}$, i.e. $\phi(\alpha) \in \bigcup[\nabla^*]_{\infty^*}$. Hence $[\nabla^*]_{\infty^*} \in \mathbf{C}(\phi(\alpha))$. (b) by (a) and Lemma 4.2(a). □

THEOREM 8.3. *Let ϕ be a continuous and bounded mereological mapping from $\mathcal{R}$ to $\mathcal{R}^*$. Then f_ϕ is a continuous function.*

Proof. Suppose $f_\phi([\nabla]_\infty) \in O^*$, where O^* is an open set in $P_{\mathcal{R}}$. By Lemma 4.10(a) there is then a region α^* in Ω^* such that $f_\phi([\nabla]_\infty) \in \mathbf{I}(\alpha^*)$ and $\mathbf{C}(\alpha^*) \subseteq O^*$. Hence $[\phi(\nabla)]_{\infty^*} \in \mathbf{I}(\alpha^*)$ by Definition 8.6. So $\alpha^* \in \nabla^*$ for every ultrafilter $\nabla^* \in [\phi(\nabla)]_{\infty^*}$, and hence $\alpha^* \in \phi(\nabla')$ for every ultrafilter $\nabla' \in [\nabla]_\infty$. By Definition 8.3, and since $\phi(\mathrm{V}\Gamma) = \mathrm{V}\{\phi(\alpha) \mid \alpha \in \Gamma\}$ (Lemma 8.1(c)), there exists a region β in Ω with $\phi(\beta) \leq \alpha^*$ and $\beta \in \nabla'$ for every ultrafilter $\nabla' \in [\nabla]_\infty$. Hence $[\nabla]_\infty \in \mathbf{I}(\beta)$. $f_\phi(\mathbf{I}(\beta)) \subseteq \mathbf{C}(\phi(\beta))$ by Lemma 8.10(b). But $\mathbf{C}(\phi(\beta)) \subseteq \mathbf{C}(\alpha^*)$ by Lemma 4.2(b). So $f_\phi(\mathbf{I}(\beta)) \subseteq O^*$. So f_ϕ is continuous. □

But not every continuous function from $\mathcal{P}_{\mathcal{R}}$ to $\mathcal{P}_{\mathcal{R}^*}$ corresponds to a mereological mapping ϕ from $\mathcal{R}$ to $\mathcal{R}^*$ as the following example shows. The points in the interval $[-3, -1]$ of $\mathcal{P}$ are mapped onto the interval $[-2, 0]$ of $\mathcal{P}^*$ and those of the interval $[+1, +3]$ onto $[0, +2]$, while all points in the interval $[-1, +1]$ are mapped into the one point 0 of $\mathcal{P}^*$. This function in continuous but does not correspond to any mereological mapping.



What singles out the counterparts of continuous and bounded mereological mappings from among continuous point-functions is the following condition

If C is regular-closed then $\mathrm{Cl}\, f(C)$ *is regular-closed.*

LEMMA 8.11. *Let ϕ be a continuous and bounded mereological mapping from $\mathcal{R}$ to $\mathcal{R}^*$. Then* $\mathrm{Cl}\, f_\phi(\mathbf{C}(\alpha)) = \mathbf{C}(\phi(\alpha))$.

Proof. $\mathrm{Cl}\, f_\phi(\mathbf{C}(\alpha)) = \bigcap\{\mathbf{C}(\beta^*) \mid f_\phi(\mathbf{C}(\alpha)) \subseteq \mathbf{C}(\beta^*)\}$ by Lemma 4.6. As $f_\phi(\mathbf{C}(\alpha)) \subseteq \mathbf{C}(\phi(\alpha))$ by Lemma 8.10, $\mathrm{Cl}\, f_\phi(\mathbf{C}(\alpha)) \neq \mathbf{C}(\phi(\alpha))$ only if there is a region β_1^* of Ω^* so that $f_\phi(\mathbf{C}(\alpha)) \subseteq \mathbf{C}(\beta_1^*)$ and $\mathbf{C}(\beta_1^*) \subset \mathbf{C}(\alpha)$. In that case $\beta_1^* < \phi(\alpha)$ by Lemma 4.5(c) and so $\beta_2^* = \phi(\alpha) \wedge -\beta_1^* \neq 0$. Hence by Lemma 1.3(a) there is a region $\gamma \neq 0$ such that $\gamma \ll \beta$. Hence $\gamma \leq \phi^{-1}(\phi(\alpha))$ and, by A4, $\gamma \,\circ\!\!\!\!\mid\, \alpha$. By Definition 8.1 $\phi(\gamma) \neq 0^*$ and $\phi(\gamma) \leq \phi(\phi^{-1}(\phi(\alpha)))$; hence $\phi(\gamma) \leq \phi(\alpha)$ by Lemma 8.3(a). Hence by Lemma 1.1 $\phi(\gamma) \infty^* \phi(\alpha)$, which is impossible. Hence $\mathrm{Cl}\, f_\phi(\mathbf{C}(\alpha)) = \mathbf{C}(\phi(\alpha))$. □

THEOREM 8.4. *Let ϕ be a continuous and bounded mereological mapping from $\mathcal{R}$ to $\mathcal{R}^*$ and let C be a regular-closed set of $\mathcal{P}_\mathcal{R}$. Then* $\mathrm{Cl}\, f_\phi(C)$ *is regular-closed.*

Proof. C is $\mathbf{C}(\alpha)$ for some region α in Ω by Theorem 4.3. Hence $\mathrm{Cl}\, f_\phi(C) = \mathbf{C}(\phi(\alpha))$ by Lemma 8.11 and $\mathbf{C}(\phi(\alpha))$ is a regular-closed set of $\mathcal{P}_\mathcal{R}$ by Lemma 4.7(c). □

Next it is shown that every continuous point function f for which $\mathrm{Cl}\, f(C)$ is regular-closed when C is regular-closed can be generated from a continuous and bounded mereological mapping. Needed for this are some results from point-based topology.

LEMMA 8.12. *Let f be a continuous function from $\mathcal{P}$ to $\mathcal{P}^*$, where $\mathcal{P}^*$ is a T_2 space. Then*

(a) $f(\mathrm{Cl}\, A) \subseteq \mathrm{Cl}\, f(A)$ *for any set A of $\mathcal{P}$;*
(b) $f^{-1}(C^*)$ *is closed in $\mathcal{P}$ if C^* is closed in $\mathcal{P}^*$;*
(c) $f^{-1}(O^*)$ *is open in $\mathcal{P}$ if O^* is open in $\mathcal{P}^*$;*
(d) $f(C)$ *is closed and compact in $\mathcal{P}^*$, if C is closed and compact in $\mathcal{P}$.*

THEOREM 8.5. *Let f be a continuous function from $\mathcal{P}_{\mathcal{R}}$ to $\mathcal{P}_{\mathcal{R}^*}$ such that* $\mathrm{Cl}\, f(C)$ *is regular-closed whenever C is regular-closed. Then f is f_ϕ for some continuous and bounded mereological mapping ϕ from $\mathcal{R}$ to $\mathcal{R}^*$.*

Proof. First we define a mapping ϕ in accordance with Theorem 8.1(a) as the extension of a mapping ϕ^1 defined on just the limited regions of $\mathcal{R}$. We will then show that ϕ^1 meets the restrictions $(\mathrm{i})^1$, $(\mathrm{ii})^1$ and $(\mathrm{iii})^1$ to limited regions of the clauses of Definition 8.1 and that ϕ^1 is continuous and bounded. Finally, the point-function f_ϕ generated by ϕ is shown to be identical to f.

For any limited region α in Ω, let $\phi^1(\alpha)$ be the region α^* in Ω^* such that $f(\mathbf{C}(\alpha)) = \mathbf{C}(\alpha^*)$. The definition is justified since, $\mathbf{C}(\alpha)$ being compact by Theorem 4.8, $f(\mathbf{C}(\alpha)) = \mathrm{Cl}\, f(\mathbf{C}(\alpha))$ by Lemma 8.12(d) and $\mathrm{Cl}\, f(\mathbf{C}(\alpha))$ is assumed to be regular-closed. It is easily seen that ϕ^1 meets $(\mathrm{i})^1$ and $(\mathrm{ii})^1$. As to $(\mathrm{iii})^1$, suppose β limited and $0^* \neq \alpha^* \leq \phi(\beta)$. Since $\mathbf{C}(\beta)$ is compact, $\mathrm{Cl}\, f(\mathbf{C}(\beta)) = f(\mathbf{C}(\beta))$ by Lemma 8.12(d) and so $\mathbf{C}(\phi(\beta)) = f(\mathbf{C}(\beta))$. Hence $\mathbf{I}(\alpha^*) \leq f(\mathbf{C}(\beta))$ and, since $\mathbf{I}(\alpha^*) \neq \varnothing$ by Lemma 4.4(c), $f^{-1}(\mathbf{I}(\alpha^*)) \cap \mathbf{C}(\beta) \neq \varnothing$. $f^{-1}(\mathbf{I}(\alpha^*))$ is open by Lemma 8.12(c); so $A = \mathrm{Cl}(f^{-1}(\mathbf{I}(\alpha^*)) \cap \mathbf{I}(\beta))$ is non-empty by Lemma 5.1(a) and regular-closed by Lemma 5.1(b). So A equals $\mathbf{C}(\alpha)$ for some region $\alpha \neq 0$ by Theorem 4.3 and Lemma 4.4(a). Since $A \leq \mathbf{C}(\beta)$, $\alpha \leq \beta$ by Lemma 4.5(c) and so α is limited by A7. $\mathbf{C}(\phi^1(\alpha)) = f(A)$ by stipulation and, since $f(A) \subseteq f(\mathrm{Cl}\, f^{-1}(\mathbf{I}(\alpha^*)))$, $f(A) \subseteq \mathrm{Cl}\, f(f^{-1}(\mathbf{I}(\alpha^*)))$ by Lemma 8.12(a), i.e. $f(A) \subseteq \mathbf{C}(\alpha^*)$. So, $\mathbf{C}(\phi^1(\alpha)) \subseteq \mathbf{C}(\alpha^*)$, and hence $\phi^1(\alpha) \leq \alpha^*$ by Lemma 4.5(c). So, $(\mathrm{iii})^1$ is met. And by Theorem 8.1(a) ϕ is a mereological mapping.

Suppose α and β are limited regions of $\mathcal{R}$ and $\alpha \infty \beta$. Then $\mathbf{C}(\alpha)$ and $\mathbf{C}(\beta)$ are compact by Theorem 4.8 and $\mathbf{C}(\alpha) \cap \mathbf{C}(\beta) \neq \varnothing$ by Theorem 4.7(a), $\mathbf{C}(\phi^1(\alpha)) = f(\mathbf{C}(\alpha))$ and $\mathbf{C}(\phi^1(\beta)) = f(\mathbf{C}(\beta))$ are compact by Lemma 8.12(d), and $\mathbf{C}(\phi^1(\alpha)) \cap \mathbf{C}(\phi^1(\beta)) \neq \varnothing$. Hence $\phi^1(\alpha)$ is limited by Theorem 4.8 and $\phi^1(\alpha) \infty^* \phi^1(\beta)$ by Theorem 4.7(a). Consequently ϕ is continuous and bounded.

Given that ϕ is a continuous and bounded mereological mapping Definition 8.6 yields a function f_ϕ from $\mathcal{P}_{\mathcal{R}}$ to $\mathcal{P}_{\mathcal{R}^*}$. It remains to be shown that f_ϕ equals f. So suppose, to the contrary, that $f_\phi([\nabla]_\infty) \neq f([\nabla]_\infty)$ for some point $[\nabla]_\infty$ of $\mathcal{P}_{\mathcal{R}}$. By Theorem 4.10 there is a region α^* in $\mathcal{R}^*$ such that $f_\phi([\nabla]_\infty) \in \mathbf{I}(\alpha^*)$, i.e. $[\phi(\nabla)]_{\infty^*} \in \mathbf{I}(\alpha^*)$, and $f([\nabla]_\infty) \in \mathbf{I}(-\alpha^*)$. Since $\alpha^* \in \bigcap[\phi(\nabla)]_{\infty^*}$ and ∇ is a limited ultrafilter, there is a limited region β of $\mathcal{R}$ with $\beta \in \nabla$, and hence $[\nabla]_\infty \in \mathbf{C}(\beta)$, and $\phi(\beta) \leq \alpha^*$. As β is limited, $\phi(\beta) = \phi^1(\beta)$. Consequently $\mathbf{C}(\phi^1(\beta)) \subseteq \mathbf{C}(\alpha^*)$, and so $f(\mathbf{C}(\beta)) \subseteq \mathbf{C}(\alpha^*)$. But then $f([\nabla]_\infty) \in \mathbf{C}(\alpha^*)$, which is impossible, since $\mathbf{C}(\alpha^*) \cap \mathbf{I}(-\alpha^*) = \varnothing$ by Lemma 4.2(c). So, $f_\phi([\nabla]_\infty) = f([\nabla]_\infty)$ for every point $[\nabla]_\infty$ of $\mathcal{P}_\mathcal{R}$, i.e. f_ϕ is the same function as f. □

So the bounded and continuous mereological mappings correspond to continuous point-to-point functions that preserve the topological property of being regular-closed.

The final category of mereological mappings to be considered consists of those that map 1 onto 1^* and satisfy the condition

> $\phi(\alpha) \propto^* \phi(\beta)$ *if and only if* $\alpha \propto \beta$ *and* $\phi(\alpha)$ *is limited if and only if* α *is limited.*

These *topological functions* will be shown to be 1-to-1 functions and to correspond to homeomorphisms between the associated point-based topologies.

DEFINITION 8.8. A mereological mapping ϕ is a *topological function* if (i) $\phi(1) = 1^*$, (ii) $\phi(\alpha) \propto^* \phi(\beta)$ if and only if $\alpha \propto \beta$, and (iii) $\phi(\alpha)$ is limited if and only if α is limited.

DEFINITION 8.9. Let $\mathcal{P} = \langle P, \mathbf{C} \rangle$ and $\mathcal{P}^* = \langle P^*, \mathbf{C}^* \rangle$ be point-based topologies. Then a function f from P to P^* is *homeomorphic* if f is a one-to-one function onto P^* and both f and f^{-1} are continuous.

As already indicated, topological functions turn out to be isomorphisms.

LEMMA 8.13. *If ϕ is a topological function then ϕ is a one-to-one mapping.*

Proof. Suppose $\phi(\alpha) = \phi(\beta)$ but $\alpha \neq \beta$, say $\alpha \wedge -\beta \neq 0$. Then there is a region $\gamma \neq 0$ with $\gamma \leq \alpha$ and $\gamma \not\propto \beta$ by Lemma 1.3. Then $\phi(\gamma) \neq 0$ and, on the one hand, $\phi(\gamma) \leq \phi(\alpha)$, while, on the other hand, $\phi(\gamma) \not\propto \phi(\beta)$, i.e. $\phi(\gamma) \not\propto \phi(\alpha)$. But by Lemma 1.1 this is impossible. □

It follows of course that ϕ^{-1} is also a topological function.

THEOREM 8.6. *If ϕ is a topological function from $\mathcal{R}$ to $\mathcal{R}^*$ then f_ϕ is a homeomorphism from $\mathcal{P}_\mathcal{R}$ to $\mathcal{P}_{\mathcal{R}^*}$.*

Proof. Since ϕ and ϕ^{-1} are both continuous and bounded, f_ϕ is one-to-one and continuous by Theorem 8.5 and $(f_\phi)^{-1} = f_{\phi^{-1}}$ is continuous. □

THEOREM 8.7. *If f is a homeomorphism from $\mathcal{P}_\mathcal{R}$ to $\mathcal{P}_{\mathcal{R}^*}$ then there is a topological function ϕ from $\mathcal{R}$ to $\mathcal{R}^*$ such that $f = f_\phi$.*

Proof. Let C be regular-closed in $\mathcal{P}_{\mathcal{R}}$ and C^* be regular-closed in $\mathcal{P}_{\mathcal{R}^*}$. Then $f^{-1}(C^*)$ is regular-closed in $\mathcal{P}_{\mathcal{R}}$ and, since f^{-1} is continuous, $f(C)$ is regular-closed in $\mathcal{P}_{\mathcal{R}^*}$, both by Lemma 8.12(b,c). Hence Theorem 8.5 implies that $f = f_\phi$, where ϕ is a topological function. □

As a consequence we obtain again, but from a different direction, the result arrived at the end of Section 5, namely that $\mathcal{R}$ and $\mathcal{R}^*$ are isomorphic if and only if $\mathcal{P}_{\mathcal{R}}$ and $\mathcal{P}_{\mathcal{R}^*}$ are homeomorphic.

The discussion in this section has application to first-order statements concerning space, time and perhaps other non-atomic totalities. In particular, spatial predicates and relations can be construed as mereological mappings.

One significant result of this section is that there is a whole class of mappings in the region-based sense that have no counterparts in point set topology. Various examples of this kind are of course well-known, e.g. the case of an instantaneous change from one state to another, which has given rise to the puzzling question: what is the case *at the moment of change*? The answer is of course that moments of time, as opposed to intervals, are not the primary bearers of temporal characteristics. So, while in many cases we can attribute such characteristics to a moment in a derivative sense, namely when it falls within an interval all of which has the characteristic, this possibility does not exist for the moment of change. So region-based topology is capable of describing certain spatial mappings that cannot be represented in point-based topology.

Another result of some importance is that in a *continuous* region-based topology any spatial mapping is fully described by its values for convex regions. This is obviously of interest, as the non-atomic domains we are interested in, physical space and time, are typically continuous.

APPENDIX

The constraints governing the polyadic relation of connection are a new constraint M0, obvious modifications of A2–A5 and A9, plus the axioms concerning limitedness, i.e. A6–A8.

M0 *If* $\infty(\Gamma \cup \Sigma)$, *then* $\infty\Gamma$
M2 If $\alpha \neq 0$, then $\infty\{\alpha\}$
M3 $\not\infty\{0\}$
M4 *If* $\infty(\Gamma \cup \{\alpha\})$ *and* $\alpha \leq \beta$ *then* $\infty(\Gamma \cup \{\beta\})$
M5 If $\infty(\Gamma \cup \{\alpha \vee \beta\})$ then $\infty(\Gamma \cup \{\alpha\})$ or $\infty(\Gamma \cup \{\beta\})$

M9 *If* $\infty\,\Gamma$ *then there is a set of regions* Γ^* *with* $\infty\,\Gamma^*$ *which contains for every region* $\alpha \in \Gamma$ *a limited region* $\alpha' \leq \alpha$.

With $\alpha \infty \beta$ defined as $\infty\{\alpha, \beta\}$ axioms A1–A9 can easily be derived in this system. And it is obvious that *points* should be defined as maximal connected sets of regions. In virtue of constraint M9 every such set contains at least one limited region.

DEFINITION A.1. A set of regions Γ is a *point* if and only if $\infty\,\Gamma$ and $\not\infty\,\Sigma$ for any proper superset Σ of Γ.

With the help of appropriate definitions and along similar lines as in Section 4 a locally compact point-based topology can be extracted from any given region-based topology in the present sense. But different region-based topologies may give rise to the same (modulo homeomorphism) point-based topology, as the diagram in Section 5 has shown. Conversely, given any locally compact topology, a region-based topology can be defined by taking regular-closed sets as regions, as in Section 5. But different (i.e. non-homeomorphic) point-based topologies may yield the same (modulo isomorphism) region-based topology.

As we have seen, the 2-place connection relation is insufficient for the description of topological structure when only constraints A1–A9 are at hand; the polyadic relation needs to be invoked. However, in the presence of A10 the polyadic relation is definable in terms of the dyadic ∞.

DEFINITION A.2. $\infty\,\Gamma$ is and only if there is a point $[\nabla]_\infty$ such that $\Gamma \subseteq \bigcup[\nabla]_\infty$.

Given this definition, M0 is immediate, M2 follows from Lemma 4.4(b); M3 from clause (i) of Definition 3.2 and A3; M4 from clauses (i) and (iii) and A4; M5 from clause (ii). As to M9 let α be in Γ and $\Gamma \subseteq \bigcup[\nabla]_\infty$, where ∇ is limited and $\alpha \in \nabla$. Then there is a limited region β in ∇. Hence $\alpha' = \alpha \wedge \beta$ is limited by A7, $\alpha' \leq \alpha$, and $\alpha' \in \nabla$. So for every region α in Γ there is a limited region $\alpha' \leq \alpha$ in $\bigcup[\nabla]_\infty$. Hence $\bigcup[\nabla]_\infty$ can play the role of the set Γ^* in M9, which is thereby proved.

NOTES

[1] I am most grateful to David Bostock for stimulating discussions and extremely useful advice. I also thank the referee for helpful suggestions.

[2] A *non-degenerate Boolean algebra* is a set with two distinguished elements 0 and 1, two binary operations $\wedge$ and $\vee$, and a unary operation $-$, satisfying

$$
\begin{array}{ccc}
-0 = 1 & & -1 = 0 \\
\alpha \wedge 0 = 0 & & \alpha \vee 1 = 1 \\
\alpha \wedge 1 = \alpha & & \alpha \vee 0 = \alpha \\
\alpha \wedge -\alpha = 0 & & \alpha \vee -\alpha = 1 \\
 & --\alpha = \alpha & \\
\alpha \wedge \alpha = \alpha & & \alpha \vee \alpha = \alpha \\
-(\alpha \wedge \beta) = -\alpha \vee -\beta & & -(\alpha \vee \beta) = -\alpha \wedge -\beta \\
\alpha \wedge \beta = \beta \wedge \alpha & & \alpha \vee \beta = \beta \vee \alpha \\
\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma & & \alpha \vee (\beta \vee \gamma) = (\alpha \vee \beta) \vee \gamma \\
\alpha \wedge (\beta \vee \gamma) = (\alpha \wedge \beta) \vee (\alpha \wedge \gamma) & & \alpha \vee (\beta \wedge \gamma) = (\alpha \vee \beta) \wedge (\alpha \vee \gamma)
\end{array}
$$

[3] See Bostock (1979), Ch. 2, Sec. 4.

[4] In a *complete* Boolean algebra the infinitary de Morgan identities

$$-\Lambda\Sigma = \mathrm{V}\{-\alpha \mid \alpha \in \Sigma\} \quad \text{and} \quad -\mathrm{V}\Sigma = \Lambda\{-\alpha \mid \alpha \in \Sigma\}$$

hold.

[5] See the Appendix, which deals with this case.

[6] See Whitehead (1929), Tarski (1956), and Menger (1978).

[7] Another conception of points, advocated by David Bostock, takes a point to be determined by a pair of regions which meet in that point and that point only. This relation between two regions is definable in terms of connection.

[8] A *proper filter* Φ in a Boolean algebra is a subset of the algebra such that

$$
\begin{array}{c}
1 \in \Phi, \qquad 0 \notin \Phi \\
\textit{if } \alpha \in \Phi \textit{ and } \beta \in \Phi, \textit{ then } \alpha \wedge \beta \in \Phi, \\
\textit{if } \alpha \in \Phi \textit{ and } \alpha \leq \beta, \textit{ then } \beta \in \Phi.
\end{array}
$$

An *ultrafilter* ∇ in a Boolean algebra is a maximal proper filter. It is a characteristic of ultrafilters that

$$\textit{if } \alpha \vee \beta \in \nabla, \textit{ then } \alpha \in \nabla \textit{ or } \beta \in \nabla,$$

so that in particular

$$\textit{exactly one of } \alpha \textit{ and } -\alpha \textit{ belongs to } \nabla.$$

[9] The set of ultrafilters of a Boolean algebra is known as the *Stone space* of the algebra. Taking ultrafilters as points amounts to assuming that ultrafilters are never collocated with one another, and hence that non-overlapping regions are never connected with one another; i.e. it amounts to the absence of topological structure. Not surprisingly then, the Stone space of a Boolean algebra turns out to be a totally disconnected space in the sense of point-set topology.

[10] Proof that any proper filter can be extended to an ultrafilter appeals to the axiom of choice. Cf. Sikorski (1969), p. 19.

[11] Or, dropping the proviso that the Boolean algebras of $\mathcal{R}_1$ and $\mathcal{R}_2$ are themselves complete, we have this result: If $\mathcal{P}_{\mathcal{R}_1}$ and $\mathcal{P}_{\mathcal{R}_2}$ are homeomorphic, then $\mathcal{R}_1' = \langle \Omega_1^c, \infty_1, \Delta_1 \rangle$

and $\mathcal{R}_2' = \langle \Omega_2^c, \infty_2, \Delta_2 \rangle$ are isomorphic, where Ω_1^c is a minimal completion of Ω_1 and Ω_2^c is a minimal completion of Ω_2.

[12] Useful texts are Sutherland (1975) and Halmos (1963).

[13] For an elegant approach to region-based topology, according to which points correspond to sets of regions that are maximal under polyadic connection, see Johanson (1981).

[14] Let R be $|-1,+1|$, i.e. the limited linear region between -1 and $+1$ and let β be $|1/2,1| \vee |1/8,1,4| \vee \cdots \vee |1/2^{2n+1}, 1/2^{2n}| \vee \cdots$. Let $\alpha' \infty \beta'$ obtain whenever α' and β' are connected, except that $\alpha' \not\infty \beta'$ when $\alpha' \leq \alpha = |-1,0|$ and β' is an interior region of β ($\beta' \ll \beta$). Then A1–A10 are met, but in some cases $\alpha \not\infty \beta'$, even though α and β' are infinitely close.

[15] Note that a connected topological space need not be locally connected.

[16] Tarski (1956).

[17] Whitehead (1929).

[18] It is not claimed that this definition of *limited* as, in effect, *everywhere continuous* throws much light on the concept of limitedness. A more illuminating definition can be had in the presence of B6 (instead of B4 and B5). This, however, makes use of the concept *there is just one point coincident with both α and β* (expressed of course in the language of regions without invoking points), a concept not investigated in the present paper. (See Note 6.)

[19] Roeper (1985).

REFERENCES

1. Bostock, D.: *Logic and Arithmetic, Volume 2.* Clarendon Press, Oxford, 1979.
2. Halmos, P. R.: *Lectures on Boolean Algebras.* Van Nostrand, Princeton, 1963.
3. Johanson, A. A.: 'Topology without points', *Quæstiones Mathematicæ*, **4** (1981), 185–200.
4. Menger, K.: 'Topology Without Points', in: *Selected Papers in Logic and Foundations, Didactics, Economics.* D. Reidel, Dordrecht, 1978, 80–107.
5. Roeper, P.: 'Generalisation of First-Order Logic to Nonatomic Domains', *Journal of Symbolic Logic*, **50** (1985), 815–838.
6. Sikorski, R.: *Boolean Algebras.* Springer-Verlag, Berlin, 1969.
7. Sutherland, W. A.: *Introduction to Metric and Topological Spaces*, Oxford University Press, Oxford, 1975.
8. Tarski, A.: 'Foundations of the Geometry of Solids' in: *Logic, Semantics, Meta-Mathematics.* Oxford University Press, Oxford, 1956, 24–29.
9. Whitehead, A. N.: *Process and Reality*, The Social Science Book Store, New York, 1929, 449–459.

Philosophy Department,
The Faculties,
The Australian National University,
Canberra ACT 0200,
Australia